Q11. Show that the function f in A $= R - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one − one and onto. Hence find f^{-1}.

Answer: $f(x) = \frac{4x+3}{6x-4}$

Let $f(x_1) = f(x_2)$

or $\frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4}$

or $24x_1x_2 + 18x_2 - 16x_1 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$

or $18x_2 + 16x_2 = 18x_1 + 16x_1$

or $34x_2 = 34x_1$

or $x_2 = x_1$

Since, $\frac{4x+3}{6x-4}$ is a real number, therefore, for every y in the co-domain of f, there exists a number x in $R - \left\{\frac{2}{3}\right\}$ such that $f(x) = y = \frac{4x+3}{6x-4}$

Therefore, f(x) is onto.

Hence f^{-1} exists.

Now, let $y = \frac{4x+3}{6x-4}$

or $6xy - 4y = 4x + 3$

or $6xy - 4x = 4y + 3$

or $x(6y - 4) = 4y + 3$

or $x = \frac{4y+3}{6y-4}$

or $y = \frac{4x + 3}{6x - 4}$

interchanging the variables x and y

or $f^{-1}(x) = \frac{4x+3}{6x-4}$ [putting $y = f^{-1}(x)$]

Q12. Find the value of the following:

$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right], |x|<1, y>0$ and $xy<1$.

Answer: We know that

$\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x \ for\ |x|\le 1 \quad \dots(1)$

$\cos^{-1}\frac{1-y^2}{1+y^2}=2\tan^{-1}y \ for\ y>0 \quad \dots(2)$

$\therefore \sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}=2\tan^{-1}x+2\tan^{-1}y$

$Therefore\ \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$=\tan\frac{1}{2}[2\tan^{-1}x+2\tan^{-1}y]$

$=\tan[\tan^{-1}x+\tan^{-1}y]$

$=\tan\left(\tan^{-1}\frac{x+y}{1-xy}\right) \quad \left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}, \text{for xy} <1\right]$

$=\frac{x+y}{1-xy}$

Or, Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Answer: We know that

$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}, xy<1$

We have,

$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$

$=\left[\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)\right]+\tan^{-1}\frac{1}{8}$

$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right)+\tan^{-1}\frac{1}{8}$

$=\tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\frac{1}{8}$

$=\tan^{-1}\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}=\tan^{-1}\frac{65}{65}$

$=\tan^{-1}1=\frac{\pi}{4}$

Hence, $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Q13. Using properties of determinants prove the following:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Answer: $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$\Delta = \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= (1+x+x^2)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$$\Delta = (1+x+x^2)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

$$= (1+x+x^2)(1-x)(1-x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

$$= (1-x^3)(1-x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

Expanding along R_1, we have

$$\Delta = (1-x^3)(1-x)(1)\begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix}$$

$$= (1-x^3)(1-x)(1+x+x^2)$$

$$= (1-x^3)(1-x^3)$$

$$= (1-x^3)^2$$