Q16. Show that the function $f(x) = |x-3|, x \in \mathrm{R}$, is continuous but not differentiable at $x = 3$.

Answer: We can define $f(x) = |x-3| = \begin{cases} 3-x, & x < 3 \\ x-3, & x \geq 3 \end{cases}$

Let a be a real number.

Case 1:

If $a < 3$, Then f(a) = 3 – c

$\lim_{x\to a} f(x) = \lim_{x\to a}(3-x) = 3-a$

Since, $\lim_{x\to a} f(x) = f(a)$, f is continuous at all negative real numbers.

Case 2:

If $a = 3$, Then f(a) = 3 – 3 = 0

$\lim_{x\to a} f(x) = \lim_{x\to a}(x-3) = 3-3 = 0$

Since, $\lim_{x\to c} f(x) = f(3)$, f is continuous at $x = 3$.

Case 3:

If $a > 3$, Then f(a) = a – 3.

Since, $\lim_{x\to a} f(x) = f(a)$, f is continuous at all positive real numbers.

Therefore, f is continuous function.

Now, we have to show that $f(x) = |x-3|, x \in \mathrm{R}$, is not differentiable at $x = 3$.

Considering the left hand limit of f at x = 3.

$\lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \frac{|3+h-3|-|3-3|}{h} = \lim_{h\to 0^-} \frac{|h|-0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$

Considering the right hand limit of f at $x = 3$

$\lim_{h\to 0^+} \frac{(3+h)-(3)}{h} = \lim_{h\to 0^+} \frac{|3+h-3|-|3-3|}{h} = \lim_{h\to 0^+} \frac{|h|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$

$$\lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h} \neq \lim_{h\to 0^+} \frac{(3+h)-(3)}{h}$$

Hence f is not differentiable at x = 3.

Or, If $x = a\sin t$ and $y = a\left(\cos t + \log\tan\frac{t}{2}\right)$, find $\frac{d^2y}{dx^2}$.

Answer:

$y = a\left(\cos t + \log\tan\frac{t}{2}\right)$ $y = a\left[(\cos t) + \left(\log\tan\frac{t}{2}\right)\right]$ $\frac{dy}{dt} = a\left[-\text{sint} + \cot\frac{t}{2}\times\sec^2\frac{t}{2}\times\frac{1}{2}\right]$ $\frac{dy}{dt} = a\left[-\text{sint} + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$ $\frac{dy}{dt} = a\left[-\text{sint} + \frac{1}{\text{sint}}\right] = a\left[\frac{-\sin^2 t + 1}{\text{sint}}\right]$ $\frac{dy}{dt} = a\left(\frac{\cos^2 t}{\text{sint}}\right) \to (1)$	$x = a\sin t$ $\frac{dx}{dt} = acost \to (2)$ or $\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = cott$ or $\frac{dy}{dx} = cott$ $\therefore \frac{d^2y}{dx^2} = -\text{cosec}^2\, t \times \frac{dt}{dx}$ $\frac{d^2y}{dx^2} = -\text{cosec}^2\, t \times \frac{1}{a\cos t}$ $= -\frac{1}{a\sin^2 t\cos t}$

Q17. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)}$

Answer: Let $I = \int \frac{\sin(x-a)}{\sin(x+a)}$

Let $(x + a) = t$ $then$ $x - a = t - 2a$

$\therefore I = \int \frac{\sin(t-2a)}{\text{sint}} dt$

$= \int \frac{\text{sint}\cos 2a - \text{cost}\sin 2a}{\text{sint}} dt$

$= \int(\cos 2a - \text{cott}\sin 2a)dt$

$= (\cos 2a)\,t - \sin 2a \log|\sin t| + C$

$= (x + a)\cos 2a - \sin 2a \log|\sin(x + a)| + C$

Or, Evaluate: $\int \frac{5x-2}{1+2x+3x^2}\, dx$

Answer: $\int \frac{5x-2}{1+2x+3x^2}\, dx$

$= 5\int \frac{x-\frac{2}{5}}{1+2x+3x^2}\mathrm{dx}$

$= \frac{5}{6}\int \frac{6x-\frac{12}{5}}{1+2x+3x^2}\mathrm{dx}$

$= \frac{5}{6}\int \frac{6x+2-\frac{12}{5}-2}{1+2x+3x^2}\mathrm{dx}$

$= \frac{5}{6}\int \frac{6x+2-\frac{22}{5}}{1+2x+3x^2}\mathrm{dx}$

$= \frac{5}{6}\int \frac{6x+2}{1+2x+3x^2}\mathrm{dx} - \frac{5}{6}\times\frac{22}{5}\int \frac{1}{3\left\{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right\}}\mathrm{dx}$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}}\mathrm{dx}$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{9}\times\frac{3}{\sqrt{2}}\tan^{-1}\frac{\left(x+\frac{1}{3}\right)}{\frac{\sqrt{2}}{3}} + C$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\times\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

Q18. Evaluate: $\int \frac{x^2}{(x^2+4)(x^2+9)}\, dx$

Answer:

Substituting $y = x^2$ $\frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)} = \frac{A}{y+4} + \frac{B}{y+9}$ $y = A(y+9) + B(y+4)$ $y = (A+B)y + 9A + 4B$ Comparing coefficient of both sides A + B = 1 and 9A + 4B = 0 Solving, we get $A = -\frac{4}{5}$ and $B = \frac{9}{5}$	$\therefore I = \int \left\lvert\frac{-4}{5(x^2+4)} + \frac{9}{5(x^2+9)}\right\rvert dx$ $= -\frac{4}{5}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5}\times\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$ $= -\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) + C$