Q19. Evaluate: $\int_0^4 [|x| + |x-2| + |x-4|]\ dx$

Answer: $\int_0^4 [|x| + |x-2| + |x-4|]\ dx$

We can split the limit into two parts from 0 to 2 and 2 to 4.

In 0 to 2 range $|x| + |x-2| + |x-4| = x - x + 2 - x + 4$

And 2 to 4 range $|x| + |x-2| + |x-4| = x + x - 2 - x + 4$

$= \int_0^2 f(x)\, dx + \int_2^4 f(x)\ dx$

$= \int_0^2 (x - x + 2 - x + 4)\ dx + \int_2^4 (x + x - 2 - x + 4)\ dx$

$= \int_0^2 (6 - x)\ dx + \int_2^4 (x + 2)\ dx$

$= \left[6x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} + 2x\right]_2^4$

$= [12 - 2] + [8 + 8 - 2 - 4] = 20$

Q20. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$, then prove that vector $2\vec{a} + \vec{b}$ is perpendicular to vector $\vec{b}$.

Answer: $|\vec{a} + \vec{b}| = |\vec{a}|$

$\Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{a}|^2$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = |\vec{a}|^2$

$\Rightarrow 2\vec{a}.\vec{b} + |\vec{b}|^2 = 0 \quad \rightarrow (1)$

Now, $(2\vec{a} + \vec{b}).(\vec{b}) = 2\vec{a}.\vec{b} + \vec{b}.\vec{b} = 2\vec{a}.\vec{b} + |\vec{b}|^2 = 0 \qquad Putting\ from\ (1)$

Since dot product of $2\vec{a} + \vec{b}$ and $\vec{b}$ is Zero then either vectors are zero or they are perpendicular to each other.

Thus, $2\vec{a} + \vec{b}$ is perpendicular to vector $\vec{b}$.

Q21. Find the coordinates of the point, where the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$ intersects the plane $x-y+z-5=0$. Also find the angle between the line and the plane.

Answer: The equation of the given line is $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2} \quad \rightarrow (1)$

Let $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=\lambda$

Hence any point on the given line is $(3\lambda+2, 4\lambda-1, 2\lambda+2)$.

Since this point lies on the given plane $x-y+z-5=0$ therefore it should satisfy

$3\lambda+2-(4\lambda-1)+ 2\lambda+2-5=0$

or $\lambda=0$

Hence $(3\lambda+2, 4\lambda-1, 2\lambda+2)=(\mathbf{2}, \mathbf{-1}, \mathbf{2})$

Let θ be the angle between the given line and the plane.

$\therefore \sin\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{(3\vec{\imath}+4\vec{\jmath}+2\vec{k}).(\vec{\imath}-\vec{\jmath}+\vec{k})}{\sqrt{3^2+4^2+2^2}\sqrt{1^2+1^2+1^2}}=\frac{3-4+2}{\sqrt{29}\sqrt{3}}=\frac{1}{\sqrt{87}}$

or $\theta=\sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$

Thus, the angle between the given line and the given plane is $\sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$.

Or, Find the vector equation of the plane which contains the line of intersection of the planes $\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+\right.$ *3k−4=0* and *r.2i+j−k+5=0* and which is perpendicular to the

plane $\vec{r}.\left(5\hat{\imath}+3\hat{\jmath}-6\hat{k}\right)+8=0$.

Answer: The equations of the given planes are:

$\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)-4=0 \quad \rightarrow (1)$

$\vec{r}.\left(2\hat{\imath}+\hat{\jmath}-\hat{k}\right)+5=0 \quad \rightarrow (2)$

The equation of the plane passing through the intersection of the planes (1) and (2) is

$\left[\vec{r}.\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)-4\right]+\lambda\left[\vec{r}.\left(2\hat{\imath}+\hat{\jmath}-\hat{k}\right)+5\right]=0$

or $\vec{r}.\left[(1+2\lambda)\hat{\imath}+(2+\lambda)\hat{\jmath}+(3-\lambda)\hat{k}\right]=4-5\lambda \quad \rightarrow (3)$

Given that plane (3) is perpendicular to the plane $\vec{r}.\left(5\hat{\imath}+3\hat{\jmath}-6\hat{k}\right)+8=0$.

$(1+2\lambda)\times 5+(2+\lambda)\times 3+(3-\lambda)\times(-6)=0$

or $19\lambda-7=0$

or $\lambda = \frac{7}{19}$.

Putting $\lambda = \frac{7}{19}$ in (3), we get

$\vec{r}.\left[\left(1+\frac{14}{19}\right)\hat{\imath}+\left(2+\frac{7}{19}\right)\hat{\jmath}+\left(3-\frac{7}{19}\right)\hat{k}\right]=4-\frac{35}{19}$

or $\vec{r}.\left(\frac{33}{19}\ \hat{\imath}+\frac{45}{19}\ \hat{\jmath}+\frac{50}{19}\ \hat{k}\right)=\frac{41}{19}$

9r $\vec{r}.\left(33\hat{\imath}+45\hat{\jmath}+50\hat{k}\right)=41$

This is the equation of the required plane.

Q22. A speaks truth in 60% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact? In the cases of contradiction do you think, the statement of B will carry more weight as he speaks truth in more number of cases than A?

Answer: Let P (A)= probability that A speak truth =$\frac{60}{100}=\frac{3}{5}$

P (B) = probability that A speak truth=$\frac{90}{100}=\frac{9}{10}$

A and B can contradict in stating a fact when one is speaking the truth and other is not speaking the truth.
Case 1: A is speaking the truth and B is not speaking the truth.

Required probability $= \mathrm{P(A)} \times \left(1-\mathrm{P(B)}\right)=\frac{3}{5}\times\left(1-\frac{9}{10}\right)=\frac{3}{50}$.

Case 2: A is not speaking the truth and B is speaking the truth.

Required probability $= \left(1-\mathrm{P(A)}\right)\times \mathrm{P(B)}=\left(1-\frac{3}{5}\right)\times\frac{9}{10}=\frac{9}{25}$.

$\therefore$ Percentage of cases in which they are likely to contradict in stating the same fact

$=\left(\frac{3}{50}+\frac{9}{25}\right)\times 100\% = \left(\frac{3+18}{50}\right)\times 100\% = 42\%$

From case 1, it is clear that it is not necessary that the statement of B will carry more weight as he speaks truth in more number of cases than A.

Q23. A school wants to award its students for the values of Honesty, Regularity and Hard work with a total cash award of Rs 6,000. Three times the award money for hard work added to that given for honesty amounts to Rs 11,000. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and Hard work, suggest one more value which the school must include for awards.

Answer:

Let x= the award money given for honesty
y= award money for regularity and
z= award money for hard work.
Since total cash award is Rs. 6,000.

$$\therefore x + y + z = 6000 \rightarrow (1)$$

Also as given three times the award money for hard work and honesty amounts to Rs. 11000.

$$\therefore 3z + x = 11000$$
$$\text{or } x + 0 \times y + 3z = 11000 \rightarrow (2)$$

Award money for honesty and hard work is double that given for regularity.

$$x + z = 2y$$
$$\text{or } x - 2y + z = 0 \rightarrow (3)$$

The above system of equations can be written in matrix form AX = B , where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$|A| = 1(0+6) - 1(1-3) + 1(-2-0) = 6 \neq 0$$

Here A is non-singular so invertible.

$$\text{Adj A} = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{Adj A}) = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + 0 - 0 \\ -12000 + 33000 - 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

Hence, x = 500, y = 2000, and z = 3500.

Thus, award money given for honesty, regularity and hard work is Rs. 500, Rs. 2000 and Rs. 3500 respectively.

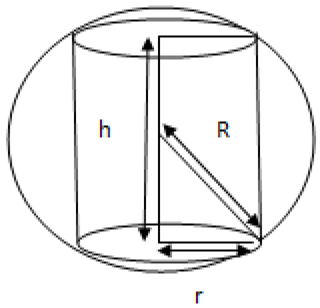
One more value school can add is disciplined.

Q24. Show that the height of the cylinder of maximum volume, which can be inscribed in a sphere of radius, R is $\frac{2R}{\sqrt{3}}$, also find the maximum volume.

Answer:

Given R= radius of the sphere.

Let r and h be the radius and the height of the inscribed cylinder respectively.



From geometry $h = 2\sqrt{R^2 - r^2}$

Let the volume of cylinder = V

$V = \pi r^2 h$

$= \pi r^2 \times 2\sqrt{R^2 - r^2}$

$= 2\pi r^2\sqrt{R^2 - r^2}$

Differentiating $V = 2\pi r^2\sqrt{R^2 - r^2}$ w.r.to r, we have

$\frac{dV}{dr} = 4\pi r\left(\sqrt{R^2 - r^2}\right) - \frac{4\pi r^3}{2\sqrt{R^2-r^2}}$

$= \frac{4\pi r(R^2-r^2)-4\pi r^3}{2\sqrt{R^2-r^2}}$

$\frac{dV}{dr} = \frac{4\pi rR^2-4\pi r^3-2\pi r^3}{2\sqrt{R^2-r^2}}$

$= \frac{4\pi rR^2-6\pi r^3}{2\sqrt{R^2-r^2}}$

For maxima or minima, $\frac{dV}{dr} = 0$ or $4\pi rR^2 - 6\pi r^3 = 0$

or $4\pi rR^2 = 6\pi r^3$

or $r^2 = \frac{2R^2}{3}$

$\frac{dV}{dr} = \frac{4\pi rR^2-6\pi r^3}{2\sqrt{R^2-r^2}}$

Now,

$\frac{d^2V}{dr^2} = \frac{1}{2}\left[\frac{\sqrt{R^2-r^2}\,(4\pi R^2-18\pi r^2)-(4\pi rR^2-6\pi r^3)\frac{(-2r)}{2\sqrt{R^2-r^2}}}{(R^2-r^2)}\right]$

$= \frac{1}{2}\left[\frac{(R^2-r^2)(4\pi R^2-18\pi r^2)+r(4\pi R^2-6\pi r^3)}{(R^2-r^2)^{\frac{3}{2}}}\right]$

$= \frac{1}{2}\left[\frac{4\pi R^4-22\pi r^2R^2+12\pi r^4+4\pi r^2R^2}{(R^2-r^2)^{\frac{3}{2}}}\right]$

Now, when $r^2 = \frac{2R^2}{3}$, $\frac{d^2V}{dr^2} < 0$.

$\therefore$ Volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When $r^2 = \frac{2R^2}{3}$, $h = 2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$

Hence, the volume of the cylinder is maximum when the height of the cylinder is $\frac{\mathbf{2R}}{\sqrt{\mathbf{3}}}$.

Or, Find the equation of the normal at a point on the curve $x^2 = 4y$ which passes through the point (1, 2). Also, find the equation of the corresponding tangent.

Answer: The equation of the given curve is $x^2 = 4y$.

Differentiating w.r.to x, we get

$\frac{dy}{dx} = \frac{x}{2}$

Let (h, k) be the coordinates of the point of contact of the normal to the curve $x^2 = 4y$.

Now, slope of the tangent at (h, k) is given by

$\left(\frac{dy}{dx}\right)_{(h,\ k)} = \frac{h}{2}$

Hence, slope of the normal at (h, k) = $-\frac{1}{\frac{dy}{dx}} = \frac{-2}{h}$

Therefore, the equation of normal at (h, k) is

$y - k = \frac{-2}{h}\ (x - h)\ \ \rightarrow (1)$

Since, it passes through the point (1, 2), we get

$2 - k = \frac{-2}{h}\ (1 - h)$

$or,\ \ k = 2 + \frac{2}{h}(1 - h)\ \ \rightarrow (2)$

Now, (h, k) lies on the curve $x^2 = 4y$, we have

$h^2 = 4k\ \ \ \rightarrow (3)$

Solving (2) and (3), we get

$h = 2\ and\ k = 1.$

From (i), the required equation of the normal is

$y - 1 = \frac{-2}{2}\ (x - 2)\ \ or, x + y = 3$

Also, slope of the tangent = 1

$\therefore$ Equation of tangent at $(1, 2)$ is $y - 2 = 1(x - 1)$

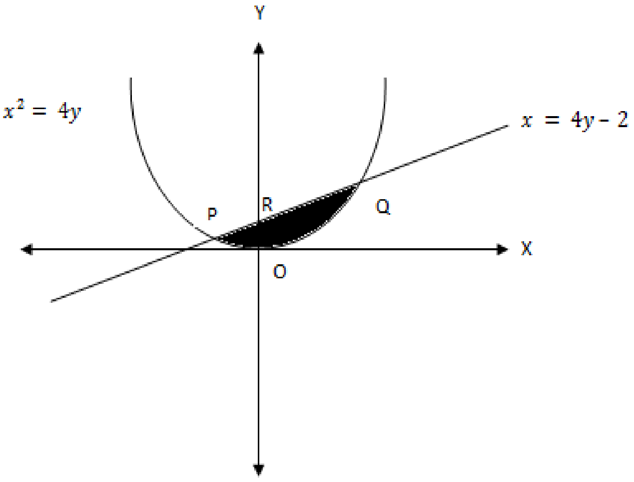
or, $y = x + 1$

Q25. Using integration, find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Answer:

$x^2 = 4y \rightarrow (1)$ is parabola with vertex as origin and axis as Y axis,
$x = 4y - 2 \rightarrow (2)$ Is straight line.

Let the straight line cut the parabola at P and Q as shown therefore the shared region OPRQO is the required area.



Solving (1) and (2):

$x = 4\left(\frac{x^2}{4}\right) - 2$
$or\ x^2 - x - 2 = 0$
$or\ x^2 - 2x + x - 2 = 0$
$or\ x(x-2) + 1(x-2) = 0$
$or\ x = -1, +2,$
$when\ x = -1, y = \frac{1}{4},$
$when\ x = 2, y = 1$
Therefore $P\left(-1, \frac{1}{4}\right)\ and\ Q(2,1)$

$\text{Area OQPO} = \text{Area OQRO} + \text{Area OPRO} \rightarrow (3)$

To find OQRO:

$= \int_0^2 \frac{x+2}{4}\, dx - \int_0^2 \frac{x^2}{4}\, dx$

$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$

$= \frac{1}{4}[2+4] - \frac{1}{4}\left[\frac{8}{3}\right]$

$= \frac{3}{2} - \frac{2}{3} = \frac{9-4}{6} = \frac{5}{6} \rightarrow (4)$

To find area OPRO:

$\int_{-1}^0 \frac{x+2}{4}\, dx - \int_{-1}^0 \frac{x^2}{4}\, dx$

$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^0 - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^0$

$= \frac{1}{4}\left[-\frac{(-1)^2}{2} - 2(-1)\right] - \frac{1}{4}\left[-\left(\frac{(-1)^3}{3}\right)\right]$

$= \frac{1}{4}\left[-\frac{1}{2} + 2\right] - \frac{1}{4}\left(\frac{1}{3}\right)$

$= \frac{3}{8} - \frac{1}{12} = \frac{7}{24} \rightarrow (5)$

Putting values from (4) and (5) in equation (3) we get

$\text{Area OQPO} = \frac{5}{6} + \frac{7}{24} = \frac{27}{24} = \mathbf{\frac{9}{8}}$ **Square Unit**

Or, Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$

Answer:

Given two circles are

$x^2 + y^2 = 4 \rightarrow (1)$ Equation of circle with centre at (0, 0) and radius 2.

$(x-2)^2 + y^2 = 4 \rightarrow (2)$ Equation of circle with centre at (2, 0) and radius 2.

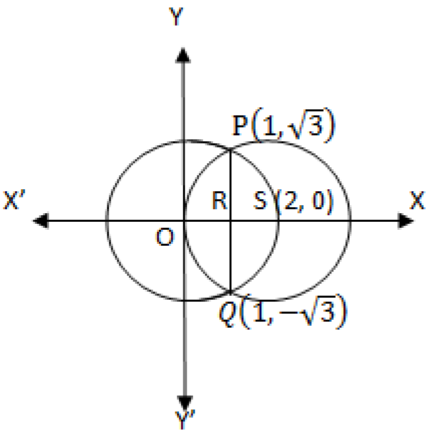
Solving equation (1) and (2), we get

$(x-2)^2 + y^2 = x^2 + y^2$

$x^2 - 4x + 4 + y^2 = x^2 + y^2$

or $x = 1$.

This gives $y = \pm\sqrt{3}$



Thus, the points of intersection of the given circles are $P(1, \sqrt{3})$ and $Q(1, -\sqrt{3})$ as shown in the figure.

Required area
= Area of the region OPSAQO
= 2[area of the region ORSPO]
= 2[area of the region ORPO+area of the region RSPR]

$= 2\left[\int_0^1 y\ dx + \int_1^2 y\ dx\right]$

$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\ dx + \int_1^2 \sqrt{4-x^2}\ dx\right]$

$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 +$

$2\left[\frac{1}{2}\ x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$

$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 +$

$\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$

$= \left[\left(-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(-1)\right] +$

$\left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$

$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$

$= \left[-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right] + \left[2\pi - \sqrt{3} - \frac{2\pi}{3}\right]$

$= \frac{8\pi}{3} - 2\sqrt{3}$