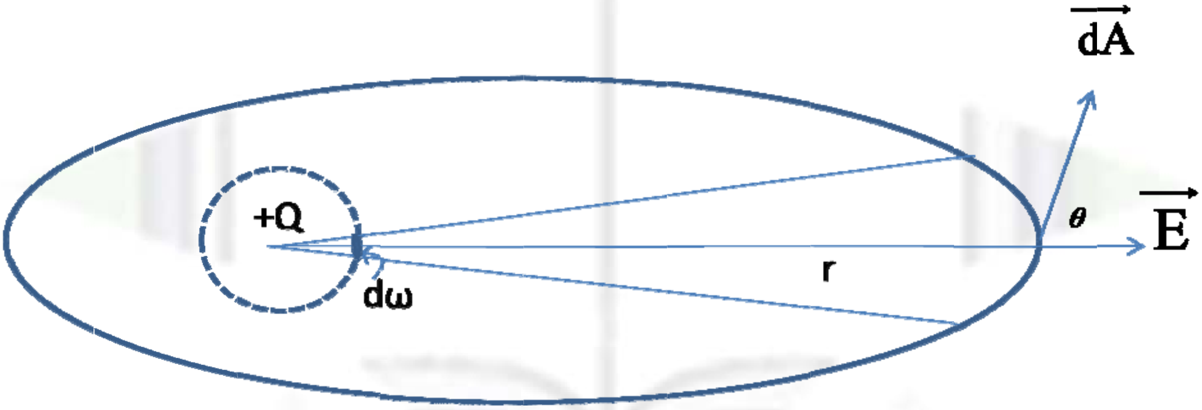
Gauss's Theorem:

Statement: "The flux through a CLOSED surface of any size and any shape is always equal to 1/ε times the charge enclosed by the surface. Where ε is the permittivity of the medium surrounding the charge. The closed surface may be real or imaginary and is known as Gaussian surface."

$$\textit{Mathematical Statement:} \varphi = \oint d\varphi = \oint \overrightarrow{E}.\overrightarrow{dA} = \frac{Q}{\varepsilon}$$

Proof: Let us consider a CLOSED surface of any size and any shape as shown. Let us consider a charge +Q at any point O inside the closed surface.

ε =Permittivity of the medium inside the surface.



Consider any point P on the surface, join OP , let OP = r

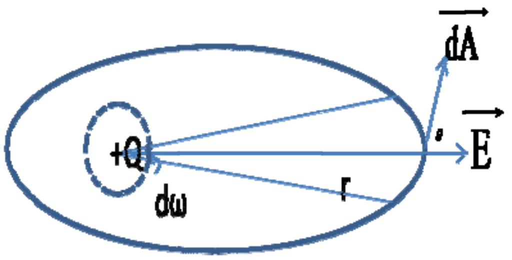
θ = the angle between vectors E and dA

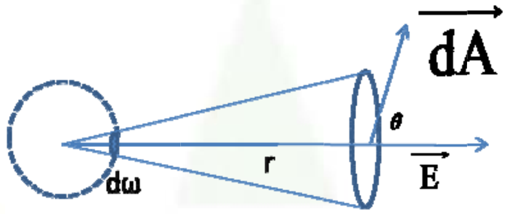
$$\text{Using the definition intensity at P}: \mathrm{E} = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2} \text{ along } \overrightarrow{\mathrm{OP}} \rightarrow (1)$$

Using definition of flux the flux through the elementry surface at P

$$\mathrm{d}\phi = \vec{\mathrm{E}}.\overrightarrow{dA} = EdA\cos\theta \rightarrow (2)$$

$$\text{Putting (1) in (2)}: \mathrm{d}\phi = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2} dA\cos\theta \rightarrow (3)$$





Let $d\omega$ be the solid angle subtended at O along OP

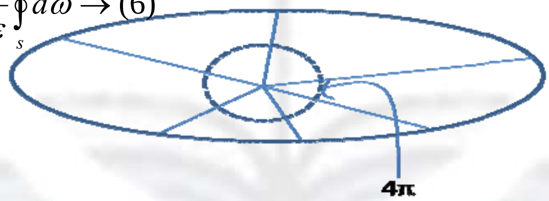
$$d\omega = \frac{dA\cos\theta}{r^2} \rightarrow (4)$$

Putting equation(4) in (3)

$$d\phi = \frac{Q}{4\pi\varepsilon} d\omega \rightarrow (5)$$

Hence the total flux through the entire closed surface can be obtained by integrating equation(5)

$$\phi = \oint_s d\phi = \oint_s \frac{Q}{4\pi\varepsilon} d\omega = \frac{Q}{4\pi\varepsilon} \oint_s d\omega \rightarrow (6)$$



$\omega = \oint_s d\omega$ = The solid angle subtended by a closed surface at any internal point

$=$ The total imaginary surface of unit radius $= 4\pi.1^2 = 4\pi \rightarrow (7)$

Putting equation (7) in (6): $\phi = \frac{Q}{4\pi\varepsilon}.4\pi = \frac{Q}{\varepsilon} \rightarrow (8)$ Proved.

Remark (1) If the enclosed charge is negative the flux through the surface is also negative

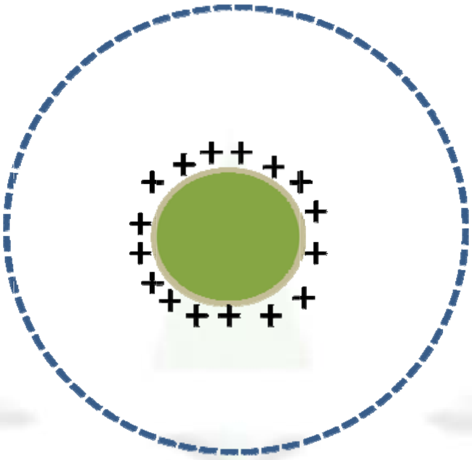
$\phi = -\frac{Q}{\varepsilon} \rightarrow$ The negative flux indicates that lines of force enter into the surface.

(2) If the charge lies outside the surface enclosed charge is zero hence flux is zero.

$\phi = 0$

Applications of Gauss's Theorem:

1. To find intensity at a point due to a uniformly charged spherical shell



Q= charge on the surface of the shell

R=Radius of the shell

r=OP=distance of the given point from the centre of the shell.

ε=Permittivity of the medium

Let E= Electric intensity at P due to the charged shell.

$$E = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2} \rightarrow (1)$$

Equation (1) is applicable when the given charge is point charge but in this case since the charged shell cannot be treated as point charge we cannot apply this formula.

Let us imagine a concentric shell of radius r passing through P this is our Gaussian surface.

Surface area of the Gaussian surface A= $4\pi r^2$

Since the lines of force leave the surface of a charged body perpendicularly hence the lines of force are all radial to the charged sphere and hence force are all radial to the charged sphere and hence they are also radial to the Gaussian surface and cutting the surface perpendicularly.

Using definition of flux through the Gaussian surface $\phi=EA=E.4\pi r^2 \longrightarrow (1)$

Case I: Let the point P lie outside the charged shell OP = r > R

Applying the Gauss's theorem the flux through the Gaussian surface

$$\phi = \frac{\text{Charged enclosed}}{\varepsilon} = \frac{Q}{\varepsilon} \rightarrow (2)$$

Equating (1) and (2):

$$E.4\pi r^2 = \frac{Q}{\varepsilon}$$

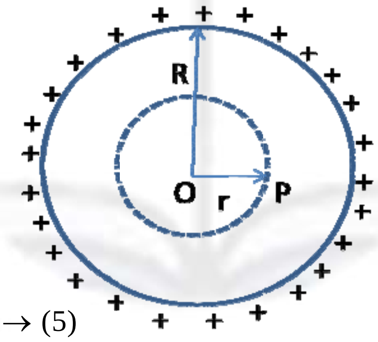
$$E = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2} \rightarrow (3)$$

If we imagine a point charge Q at the center of the shell intensity at P

$$E = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2} \rightarrow (4)$$

Thus comparing equation (3) & (4) we can say that to find potential and intensity at any external point due to a charged shell the shell can be assumed to be concentrated at its centre.

Case II: Let the point P lie inside the charged shell i.e. OP=r and r <R applying Gauss theorem the flux through the Gaussian surface



$$\phi = \frac{\text{Charged enclosed}}{\varepsilon} = \frac{0}{\varepsilon} = 0 \rightarrow (5)$$

From equation (1) and (5):

$$E.4\pi r^2 = 0$$

$$\because r \neq 0 \therefore E = 0$$

Intensity at a point inside a charged shell is zero.

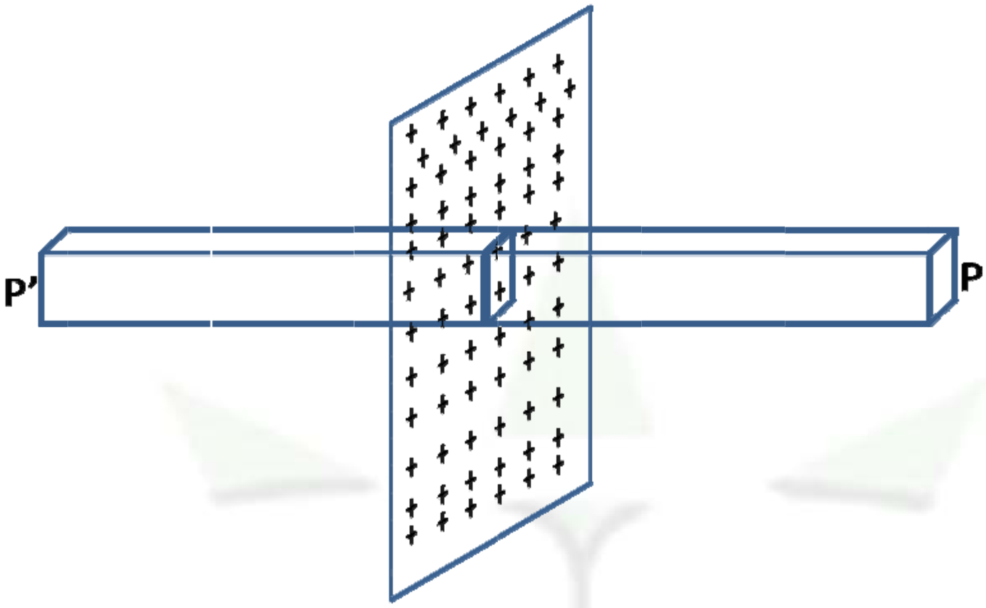
$$E = -\frac{dv}{dr} = 0$$

$$\frac{dv}{dr} = 0$$

$$v = \text{constant}$$

Thus the electric potential at every point inside a charged shell is constant and is equal to the potential on the surface of shell. Hence potential difference between two points inside a charged shell is zero. Hence no work is required to move a charged particle inside a charged shell.

(2) Intensity at a point near an infinitely charged insulating plate



Given: σ = surface density of charge i.e. charge per unit surface area

ε = permittivity of the surrounding medium

P is the given point close to the surface of the charged plate.

Let E be the intensity at P ?

Let us imagine a point P' opposite to P and at the same distance at P.

Let us imagine a rectangular box passing through the points P & P'. This closed surface is our Gaussian surface and is known as pill box.

Since the lines of force are emitted normally from the surface they are parallel to the two vertical and two horizontal faces of the Gaussian surface and hence flux through this faces will be zero. The lines of force are perpendicular to the two end faces only.

Hence the total flux through the Gaussian surface

$$\phi = 0+0+0+0+\text{EA}+\text{EA} = 2\text{EA} \rightarrow (1)$$

Applying Gauss's theorem the flux through the Gaussian surface

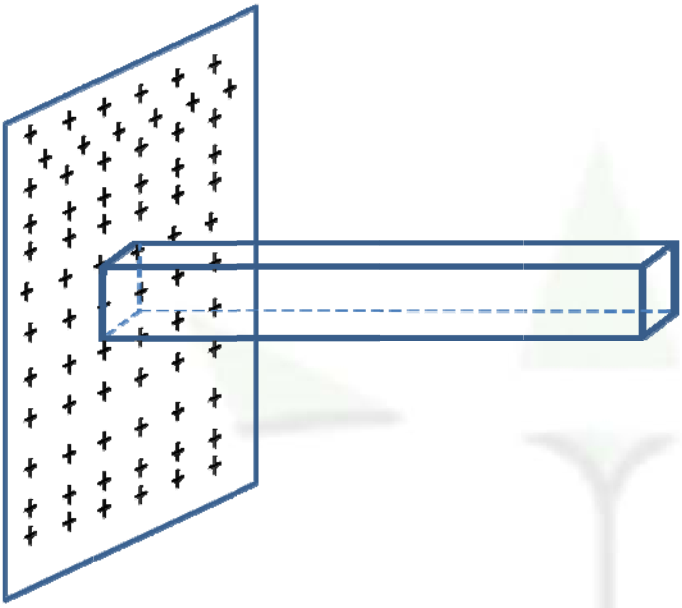
$$\phi = \frac{\text{charged enclosed}}{\text{permittivity}} = \frac{\sigma A}{\varepsilon} \rightarrow (2)$$

From equation (1) and (2)

$$2\text{EA} = \frac{\sigma A}{\varepsilon}$$

$$\text{E} = \frac{\sigma}{2\varepsilon} \rightarrow (3)$$

(3) Intensity at a point near an infinitely long charged conducting plate



Since lines of force cannot exist inside a conductor we get lines of force only in the side of the charged face.

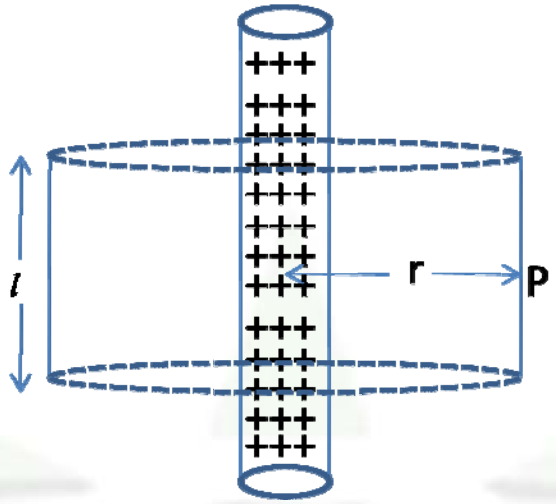
$$\phi = 0 + 0 + 0 + 0 + EA = EA \rightarrow (1)$$

$$\phi = \frac{\sigma A}{\varepsilon} \rightarrow (2)$$

$$EA = \frac{\sigma A}{\varepsilon}$$

$$E = \frac{\sigma}{\varepsilon}$$

(4) Intensity at a point due to a uniformly charged cylinder



Given:

λ = the linear density of charge i.e charge per unit length

ε = permittivity of the surrounding medium

r = distance of the given point from the axis of the cylinder

Let E = Intensity at P ?

Through P let us imagine a co-axial cylinder of radius r and length l.

This is our Gaussian surface.

Since lines of force are emitted normally from the surface of the charged body hence they are perpendicular to the curved vertical surface but are parallel to the two horizontal faces and hence flux through those faces will be zero.

$\text{Flux} = \phi = 0 + 0 + E(2\pi r l)$

$\phi = 2\pi r l E \rightarrow (1)$

Applying Gausse's theorem the flux through the Gaussian surface

$$\phi = \frac{\text{Charged enclosed}}{\text{Permittivity}} = \frac{\lambda l}{\varepsilon} \rightarrow (2)$$

From equation (1) and (2)

$$\frac{\lambda l}{\varepsilon} = 2\pi r l E$$

$$or\ E = \frac{\lambda}{2\pi\varepsilon r} \rightarrow (3)$$