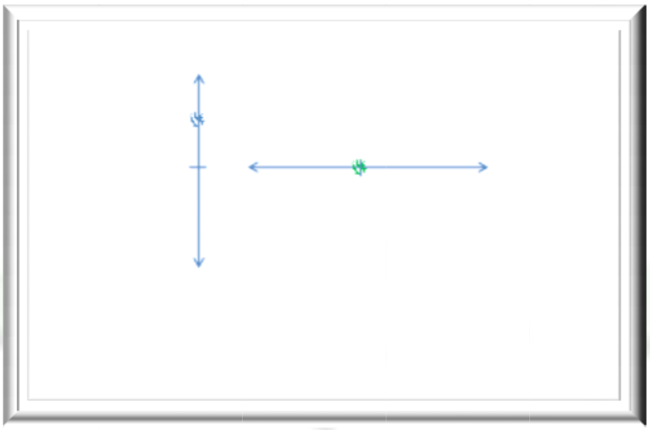
Composition of two rectangular (mutually perpendicular or orthogonal) S.H.M

Let us consider two particles executing S.H.M along any two mutually perpendicular directions say X & Y axis. The two given S.H.M have same period (T) i.e have same angular frequency ω (2π/T) but different amplitude and a phase difference between them.



Let a & b = amplitude of the two S.H.M along X and Y axis respectively.

ϕ = phase difference between the two S.H.M

Let x and y be the displacement along X and Y axis at an instant of time t.

Using displacement equation

$$x = a\sin\omega t \longrightarrow (1)$$

$$y = b\sin(\omega t \pm \phi) \longrightarrow (2)$$

$$\frac{y}{b} = \sin\omega t.\cos\phi \pm \cos\omega t.\sin\phi \longrightarrow (3)$$

$$\sin\omega t = \frac{x}{a} \longrightarrow (4)$$

$$\cos\omega t = \sqrt{1-\sin^2\omega t} = \sqrt{1-\frac{x^2}{a^2}} \longrightarrow (5)$$

Putting equation (4) and (5) in (3)

$$\frac{y}{b} = \frac{x}{a}\cos\phi \pm \sqrt{1-\frac{x^2}{y^2}}\sin\phi$$

$$\frac{y}{b} - \frac{x}{a}\cos\phi = \pm\sqrt{1-\frac{x^2}{y^2}}\sin\phi$$

$$\left(\frac{y}{b} - \frac{x}{a}\cos\phi\right) = \left(\pm\sqrt{1-\frac{x^2}{y^2}}\sin\phi\right)^2$$

$$\frac{y^2}{b^2} - 2\frac{y}{b}\frac{x}{a}\cos\phi + \frac{x^2}{a^2}\cos^2\phi = \left(1-\frac{x^2}{a^2}\right)\sin^2\phi$$

$$\frac{x^2}{a^2}\cos^2\phi + \frac{x^2}{a^2}\sin^2\phi + \frac{y^2}{b^2} - 2\frac{y}{b}\frac{x}{a}\cos\phi = \sin^2\phi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{x}{a}\frac{y}{b}\cos\phi = \sin^2\phi \longrightarrow (6)$$

Equation (6) represents the general equation of an inclined ellipse or oblique ellipse with the major and minor axes, inclined with X and Y axis. The resultant of the two given S.H.M is along this inclined ellipse with a and b as semi major and semi minor axis respectively.

Special Cases : Let the two given rectangular S.H.M be in phase

$\phi = 0 \therefore \cos\phi = \cos 0 = 1, \sin\phi = \sin 0 = 0$

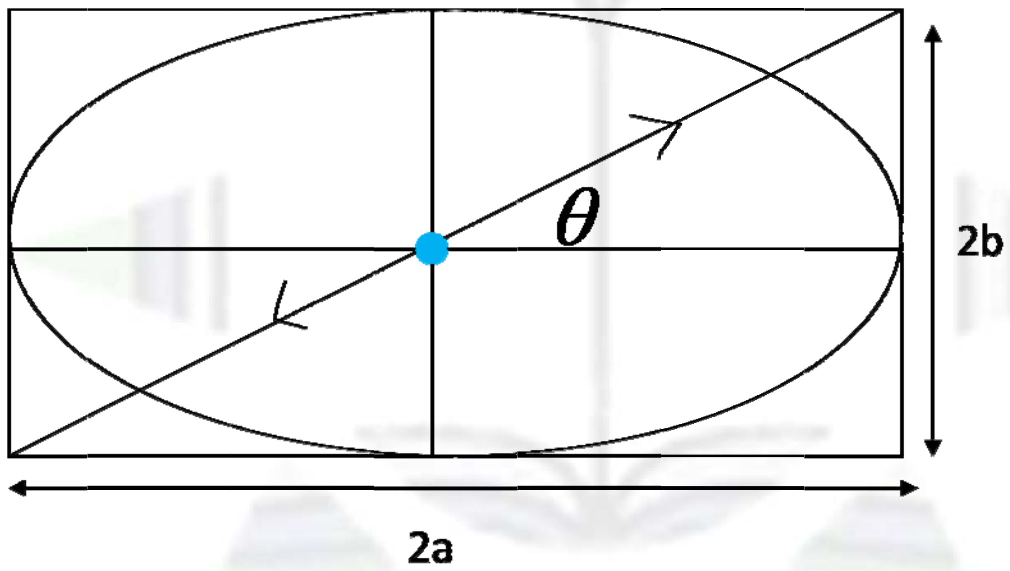
From equation (6)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{x}{a}\frac{y}{b}(1) = 0$$

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

$$\pm\left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

$$\pm y = \pm\frac{b}{a}x \longrightarrow (7)$$



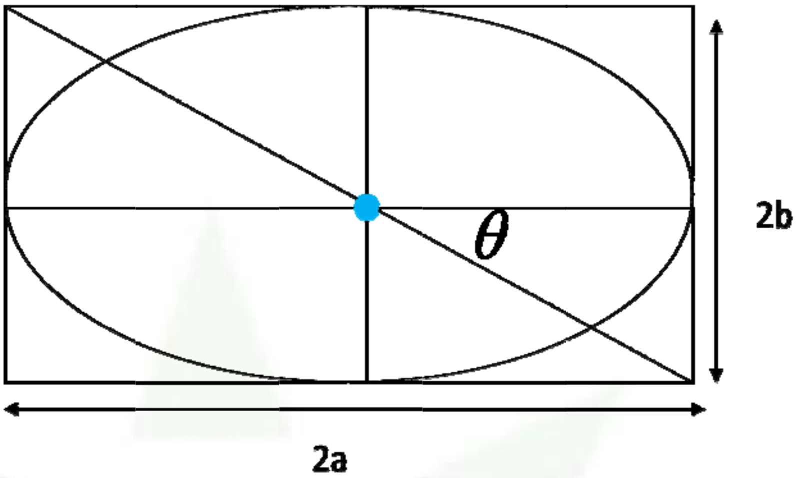
Equation (7) represents a pair of coincident straight lines, passing through the origin laying in the 1st and 3rd quadrant and inclined with the X-axis at an angle.

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

The particle oscillates along this straight line and covers the straight line twice in time period T

Case II : Let the two given S.H.M be out of phase i.e.

$$\phi = \pi$$
$$\cos\phi = \cos\pi = -1$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{x}{a}\frac{y}{b}(-1) = 0$$
$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$
$$\pm\frac{y}{b} = \mp\frac{x}{a}$$
$$\pm y = \mp\frac{b}{a}x \longrightarrow \quad (8)$$



Equation (8) represents a pair of coincident straight lines passing through the origin laying in the 2nd and 4th quadrant and inclined with the X axis at an angle.

The particle oscillates along this straight line and covers the straight line twice in time period T

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

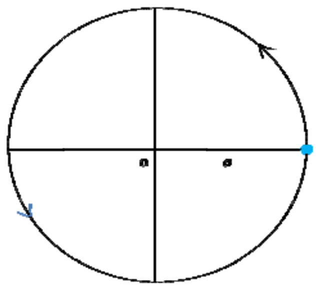
Case III : Let the phase difference

$$\phi = \frac{\pi}{2}$$
$$\therefore \cos\phi = \cos\frac{\pi}{2} = 0, \sin\phi = \sin\frac{\pi}{2} = 1$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow \quad (9)$$

Equation (9) represents a right ellipse. The particle now moves along this ellipse and covers the ellipse once in time period T.

Case IV : Phase difference

$$\phi = \frac{\pi}{2} \,\&\, a = b$$
$$\therefore \cos\phi = \cos\frac{\pi}{2} = 0, \sin\phi = \sin\frac{\pi}{2} = 1$$
$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$
$$x^2 + y^2 = a^2 \longrightarrow \quad (10)$$



Equation (10) represents a circle with center at the origin and radius is a. The particle now moves along this circle and covers the circle once in time period T.