SECTION - B

Q11. Prove the following

$$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}$$

Answer: Let $\sin^{-1}\frac{3}{5}=\alpha$ and $\cot^{-1}\frac{3}{2}=\beta$

$\Rightarrow \sin\alpha=\frac{3}{5}$ and $\cot\beta=\frac{3}{2}$

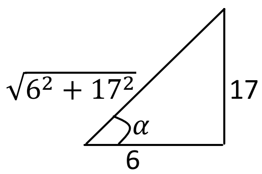
$\Rightarrow \tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}}=\frac{3}{4}$ and $\tan\beta=\frac{1}{\cot\beta}=\frac{2}{3}$

$\therefore \alpha=\tan^{-1}\frac{3}{4}$ and $\beta=\tan^{-1}\frac{2}{3}$

Thus $\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}$

$=\tan^{-1}\left[\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right]==\tan^{-1}\left(\frac{17}{6}\right)=\delta$

$\Rightarrow \tan\delta=\frac{17}{6}$



Therefore $\cos\delta=\frac{6}{\sqrt{6^2+17^2}}=\frac{6}{\sqrt{325}}=\frac{6}{5\sqrt{13}}$ $or\ \delta=cos^{-1}\left(\frac{6}{5\sqrt{13}}\right)$

Hence L. H. S. $=\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

$$=\cos\left(cos^{-1}\left(\frac{6}{5\sqrt{13}}\right)\right)=\frac{6}{5\sqrt{13}}$$

Q12. Using properties of determinants, show that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$$

Answer: L.H.S. = $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying: $R_1 \rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= 2\begin{vmatrix} b+c & c+a & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Now Applying: $R_1 \rightarrow R_1 - R_2$

$$= 2\begin{vmatrix} c & 0 & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying : $R_3 \rightarrow R_3 - R_1$

$$= 2\begin{vmatrix} c & 0 & a \\ b & c+a & b \\ 0 & c & b \end{vmatrix}$$

Applying: $R_2 \rightarrow R_2 - R_3$

$= 2\begin{vmatrix} c & 0 & a \\ b & a & 0 \\ 0 & c & b \end{vmatrix}$ Now expanding 2[c(ab-0)-b(0-ac)]=2(abc+abc)=4abc=R.H.S

Q13. Show that $f: \mathrm{N} \rightarrow \mathrm{N}$, given by

$$f(x) = \begin{cases} x + 1, \text{ if } & x \text{ is odd} \\ x - 1, \text{ if } & x \text{ is even} \end{cases}$$

Is both one-one and onto.

Answer:

<table>
<tr>
<td rowspan="6">To Show Onto
Here, $f: \mathrm{N} \rightarrow \mathrm{N}$ such that
$$f(x) = \begin{cases} x + 1, \text{ if } & x \text{ is odd} \\ x - 1, \text{ if } & x \text{ is even} \end{cases}$$
Clearly *f* is a function.
For any : $\mathrm{y} \in \mathrm{N}$ (= codomain), y can be even or odd.
When y is odd, y+1 is even, so
f (y+1) = (y+1) - 1 = y
When y is even, y-1 is odd, so
f (y-1) = (y-1) + 1 = y
$\Rightarrow f: \mathrm{N} \rightarrow \mathrm{N}$ is onto.</td>
<td>let $\mathrm{x}, \mathrm{y} \in \mathrm{N}$ s.t. $f(x) = f(y)$
We shall show: x = y</td>
</tr>
<tr>
<td>**Case I:** x and y are both even:
$f(x) = f(y) \Rightarrow \mathrm{x} - 1 = \mathrm{y} - 1 \Rightarrow \mathrm{x} = \mathrm{y}$</td>
</tr>
<tr>
<td>**Case II:** x and y are both odd:
$f(x) = f(y) \Rightarrow \mathrm{x} + 1 = \mathrm{y} + 1 \Rightarrow \mathrm{x} = \mathrm{y}$</td>
</tr>
<tr>
<td>**Case III:** x is odd and y is even
$f(x) = f(y) \Rightarrow \mathrm{x} + 1 = \mathrm{y} - 1$
$\Rightarrow \mathrm{y} - \mathrm{x} = 2$
If x is odd and y is even then y-x is odd but above relation shows y-x is 2 which is even this is not (N) possible.</td>
</tr>
<tr>
<td>**Case IV:** Similarly x is even and y is odd is also not possible(N).
$f(x) = f(y) \Rightarrow \mathrm{x} = \mathrm{y}$
$\Rightarrow f$ is one − one.
Hence *f* is both one-one and onto.</td>
</tr>
</table>

Or, Consider the binary operation
$*: R \times R \to R$ and $o: R \times R \to R$ defined as $a * b = |a - b|$ and $a \text{ o } b = a$ for all $a, b \in R$. Show that $' * '$ is commutative but not associative, $'o'$ is associative but not commutative.

Answer: $*: R \times R \to R$ is given by

$a * b = |a - b| \quad \forall\, a, b \in R$

and $o: R \times R \to R$ is given by

$a \text{ o } b = a \quad \forall\, a, b \in R$

Now $b * a = |b - a| = |a - b|$

$[\because\ |-x| = |x| \ \forall\, x \in R]$

$= a * b \quad \forall\, a, b \in R$

$\Rightarrow \star$ is a commutative binary operation on R.

To validate $(a * b) * c$ is associative or not considering arbitrary values say for a = 2, b = 4, c = 5

$(a * b) * c = (2 * 4) * 5 = |2 - 4| * 5$

$= 2 * 5 = |2 - 5| = 3$

and $a * (b * c) = 2 * (4 * 5) = 2 * |4 - 5|$

$= 2 * 1 = |2 - 1| = 1$

$\therefore (a * b) * c \neq a * (b * c)$

$\Rightarrow \star$ is not an associative operation on R.

To check operation $'o'$ we have $(a \text{ o } b) \text{ o } c = a \text{ o } c = a$

and $a \text{ o } (b \text{ o } c) = a \text{ o } b = a \quad \forall\, a, b, c \in R$

$\Rightarrow (a \text{ o } b) \text{ o } c = a \text{ o } (b \text{ o } c) \quad \forall\, a, b, c \in R$

$\Rightarrow$ o is an associative binary operation on R.

Also for a = 3, b = 2

$a \text{ o } b = 3 \text{ o } 2 = 3$

$b \text{ o } a = 2 \text{ o } 3 = 2$

$\Rightarrow a \text{ o } b \neq b \text{ o } a \quad \forall\, a, b \in R$

$\Rightarrow$ o is not a commutative operation on R.

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Q14. *If* $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Answer: Given $x = \sqrt{a^{\sin^{-1} t}},\ y = \sqrt{a^{\cos^{-1} t}}$

$\Rightarrow \log x = \log \sqrt{a^{\sin^{-1} t}} = \frac{1}{2} \sin^{-1} t \ \log a$

Differentiating with respect to t,

$\frac{1}{x} \frac{dx}{dt} = \frac{1}{2} \log a . \frac{1}{\sqrt{1-t^2}} \ldots \ldots \ldots \ldots (1)$

Again, $\log y = \log \sqrt{a^{\cos^{-1} t}} = \frac{1}{2} \cos^{-1} t \ \log a$

Differentiating with respect to t,

$\frac{1}{y} \frac{dy}{dt} = \frac{1}{2} \log a . \frac{-1}{\sqrt{1-t^2}} \ldots \ldots \ldots \ldots (2)$

Dividing equation(2) by equation (1) we get

$\frac{\frac{1}{y}\frac{dy}{dt}}{\frac{1}{x}\frac{dx}{dt}} = -1 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

Or, Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x.

Answer: Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Replacing $x = \tan \theta$

$$\therefore \frac{\sqrt{1 + x^2} - 1}{x} = \frac{\sqrt{1 + \tan^2 \theta - 1}}{\tan \theta} = \frac{\sec \theta - 1}{\tan \theta} = \frac{\frac{1}{cos\theta} - 1}{\frac{sin\theta}{cos\theta}}$$

$= \frac{1-\cos \theta}{\sin \theta} = \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}$

$= \tan \frac{\theta}{2}$

$\therefore y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right] = \tan^{-1}\left(\tan \frac{\theta}{2}\right) = \frac{1}{2} \theta = \frac{1}{2} \tan^{-1} x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{1}{1+x^2}$

Q15. If $x = a(cos\,t + t\,sin\,t)$ and $y = a\,(sin\,t - t\,cos\,t), 0 < t < \frac{\pi}{2}$,

find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

Answer: Given $x = a(cos\,t + t\,sin\,t)$

$\Rightarrow \frac{dx}{dt} = a(-sin\,t + 1.sin\,t + t\,cos\,t)$

$\Rightarrow \frac{dx}{dt} = a.t\,cos\,t \ldots\ldots\ldots\ldots.(1)$

$\Rightarrow \frac{d^2x}{dt^2} = a\,(1.cos\,t - t.sin\,t)$

$\boldsymbol{\frac{d^2x}{dt^2} == a(\cos t - t\sin t)}$

Also $y = a\,(sin\,t - t\,cos\,t)$

$\Rightarrow \frac{dy}{dt} = a(cos\,t - 1.cos\,t + t\,sin\,t)$

$\Rightarrow \frac{dy}{dt} = a.t\,sin\,t \ldots\ldots\ldots\ldots(2)$

$\Rightarrow \frac{d^2y}{dt^2} = a\,(1.sin\,t + t.\cos t)$

$\boldsymbol{\frac{d^2y}{dt^2} = a(\sin t + t\cos t)}$

Now $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{a.t\,sin\,t}{a.t\,cos\,t}$ $\quad [From\ equation\ (1) and\ (2)]$

$= \tan t$

$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\tan t$

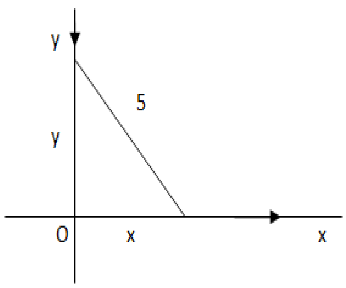
$= \sec^2 t\ .\frac{dt}{dx} = \sec^2 t/\frac{dx}{dt}$

$\boldsymbol{\frac{d^2y}{dx^2} = \sec^2 t/a.t\cos t = \frac{1}{a\,t\cos^3 t} = \frac{sec^3 t}{at}}$

1. $\boldsymbol{\frac{d^2x}{dt^2} == a(\cos t - t\sin t)}$
2. $\boldsymbol{\frac{d^2y}{dt^2} = a(\sin t + t\cos t)}$
3. $\boldsymbol{\frac{d^2y}{dx^2} = \frac{sec^3 t}{at}}$

Q16. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

Answer:



Ladder is inclined to the wall as shown the mathematical relation of sliding of ladder along X axis and displacement along Y axis can be stated as

$x^2 + y^2 = 5^2 \rightarrow (1)$

Differentiating equation (1) with respect to t we get

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$\Rightarrow y \frac{dy}{dt} = -x \frac{dx}{dt}$

$\Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} \rightarrow (2)$

When $x = 4$ m; $y^2 = 5^2 - x^2 = 5^2 - 4^2 = 3^2$

$\Rightarrow y = 3\text{m}$ and $\frac{dx}{dt} = 2\ cm/sec$

Therefore when x=400 cm, y=300 cm and

$\frac{dx}{dt} = 2\ cm/sec$ Putting these values in equation(2)

$\therefore \frac{dy}{dt} = -\frac{400}{300} \times 2 = -\frac{8}{3}\ cm/sec.$

The height of the ladder on the wall is

decreasing at rate of $\frac{8}{3}\ cm/sec.$

Negative sign simply indicating decreasing rate.