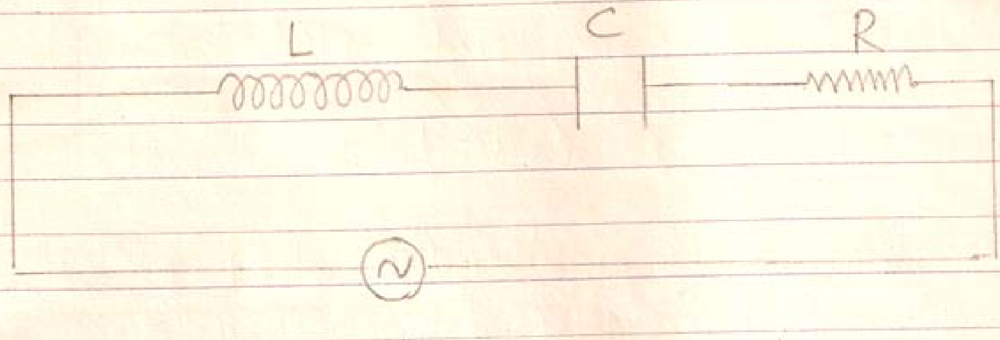
L-C-R circuit:



An alternating source of e.m.f is connected across an L-C-R circuit.

Given:

L = Self inductance of the coil.

C = The capacitance of the capacitor.

R = The resistance in the circuit.

$e = E_p \sin\omega t$ —① The instantaneous e.m.f applied across the circuit.

We first solve the circuit by e.m.f eqn method:

Let Q be the charge stored in the capacitor at the instant of time t.

i be the current flowing in the circuit at the instant of time 't'.

The e.m.f eqn. of the circuit at the instant of time 't':

$$L\frac{di}{dt} + iR + \frac{Q}{C} = e = E_p \sin\omega t$$

Differentiating w.r.t t:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\frac{dQ}{dt} = \omega E_p \cos\omega t$$

Putting $\frac{dQ}{dt} =$

$$\therefore\ L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{c} = \omega E_p \cos\omega t \quad — ②$$

Let us assume a trial Solution of eqⁿ ②:-

$$i = I_p \sin(\omega t - \phi) \quad — ③$$ where I_p & ϕ are unknown.

Since eqⁿ ③ is a solution of eqⁿ ② it must satisfy eqⁿ ② putting eqⁿ ③ in ②:

$$L.\omega^2 I_p \left\{-\sin(\omega t-\phi)\right\} + R\omega I_p \cos(\omega t - \phi) + \frac{I_p}{c}\sin(\omega t-\phi) = \omega E_p \cos\omega t$$

$$I_p \cancel{\omega}\left[R\cos(\omega t-\phi) - \left(\omega L - \frac{1}{\omega c}\right)\sin(\omega t - \phi)\right] = \cancel{\omega} E_p \cos\omega t$$

$$\text{or}\quad I_p\left[\cos\omega t\left\{R\cos\phi + (\omega L - 1/\omega c)\sin\phi\right\} + \sin\omega t\left\{R\sin\phi - \left(\omega L - \frac{1}{\omega c}\right)\cos\phi\right\}\right] = E_p \cos\omega t \quad — ④$$

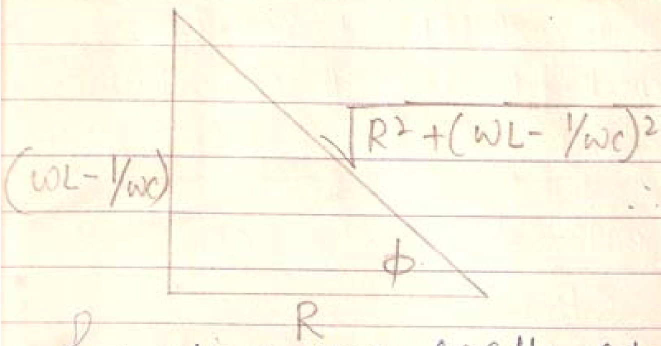
Equating the coeff. of $\sin\omega t$ from both sides of eqⁿ ④:

$$I_p\left\{R\sin\phi - \left(\omega L - \frac{1}{\omega c}\right)\cos\phi\right\} = 0$$

But $I_p \neq 0$ $\therefore$ $R\sin\phi = \left(\omega L - \frac{1}{\omega c}\right)\cos\phi$

$$\text{or}\quad \tan\phi = \frac{(\omega L - 1/\omega c)}{R} \quad — ⑤$$

from ⑤ the unknown 'ϕ' can be calculated.



$$\sin\phi = \frac{\omega L - 1/\omega C}{\sqrt{R^2+(\omega L - 1/\omega C)^2}} \quad \text{--- (6)}$$

$$\therefore \cos\phi = \frac{R}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}} \quad \text{--- (7)}$$

Equating the coeff. of $\cos\omega t$ from both sides:

$$I_p = \frac{E_p}{R\cos\phi + (\omega L - 1/\omega C)\sin\phi}$$

putting eqns (6) & (7):

$$I_p = \frac{E_p}{\dfrac{R.\ R}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}} + \dfrac{\left(\omega L-\frac{1}{\omega C}\right)\left(\omega L-\frac{1}{\omega C}\right)}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}}$$

$$\text{or } I_p = \frac{E_p\sqrt{R^2+(\omega L-1/\omega C)^2}}{R^2+(\omega L-1/\omega C)^2}$$

$$\text{or } I_p = \frac{E_p}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}} \quad \text{--- (8)}$$

From eqn. (8) we find that the total resistance offered by the L-C-R circuit to the flow of A.C i.e the impedence of the circuit

$$Z = \frac{E_p}{I_p} = \sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2} = \sqrt{R^2+(X_L-X_C)^2} \quad \text{--- (9)}$$

Phase difference between the total P.d across the circuit & the current flowing through the circuit is 'ϕ' given by

$$\tan\phi = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R} \quad \text{—— (5)}$$

Discussions: For the given values of L, C & ω we have three different possibilities.

Case I:

$$\omega L > \frac{1}{\omega C} \quad \text{i.e} \quad X_L > X_C \quad \text{or} \quad \omega > \frac{1}{\sqrt{LC}}$$

from (5) $\tan\phi = +ve$ i.e ϕ is +ve. In this case the P.d across the circuit leads the current by phase angle 'ϕ'.

Case II:

$$\omega L < \frac{1}{\omega C} \quad \text{i.e} \quad X_L < X_C \quad \text{or} \quad \omega < \frac{1}{\sqrt{LC}}$$

From eq^n (5) $\tan\phi = -ve$ i.e ϕ is negative. In this case the total P.D across the circuit lags the current by phase angle 'ϕ'.

Case III

$$\omega L = \frac{1}{\omega C} \quad \text{i.e} \quad X_L = X_C \quad \text{or} \quad \omega = \frac{1}{\sqrt{LC}}$$

From (5); $\tan\phi = 0$ i.e. $\phi = 0$. In this case the P.d across the circuit is in phase with the current i.e circuit behaves as a purely resistive circuit.

The impedence of the circuit $Z = \sqrt{R^2 + 0^2}$

$Z = R$ = Pure resistance.

In this case the impedence of the circuit is minimum & hence current flowing through the circuit is max^{m} and therefore the circuit is said to be in resonance. (current means r.m.s current).

The frequency $2\pi f = \dfrac{1}{\sqrt{LC}}$

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

This frequency is known as resonant frequency

$$\omega_r = \frac{1}{\sqrt{LC}} \quad \text{or} \quad 2\pi f_r = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

[This is an example of electrical forced vibration].

Vector diagram:

(i) Let the direction of $\vec{OX}$ represent the phase of current flowing through the circuit.

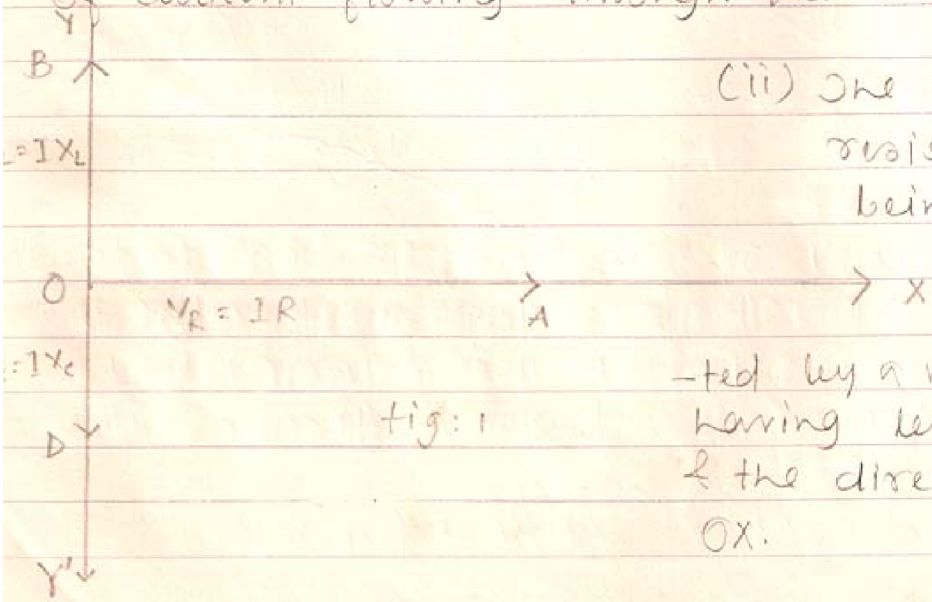


fig: 1

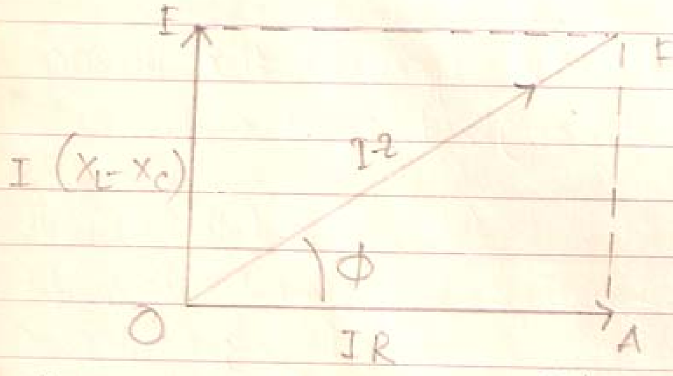
(ii) The P.D across the resistance $V_R = IR$ being in phase with current can be represented by a vector $\vec{OA}$, having length $OA = IR$ & the direction along OX.

(iii) The P.D across the inductance $V_L = I X_L$ leads the current, by phase angle $\pi/2$ & hence can be represented by a vector $\vec{OB}$ having length $OB = I X_L$ and direction along Y-axis.

(iv) P.D across the Capacitor $V_C = I X_C$ lags the current by phase angle $\pi/2$ & hence can be represented by a vector OD; having length $OD = I X_C$ & direction along OY'

Let $X_L > X_C$ ∴ $V_L > V_C$ i.e $OB > OD$

$\vec{OB}$ and $\vec{OD}$ being along the same line in opposite directions, the resultant of the two vectors will be the difference of the magnitude of the two vectors, towards the greater vector.



∴ Resultant of $\vec{OB}$ and $\vec{OD}$ is $\vec{OE}$

$OE = I(X_L - X_C)$ & direction is along OY.

The resultant of $\vec{OA}$ and $\vec{OE}$ can be found by the law of parallelogram of vectors the diagonal $\vec{OF}$ represent the value of total P.D as well as the phase difference with current.

$$OF^2 = OA^2 + AF^2 = I^2R^2 + I^2(X_L - X_C)^2$$

or $OF = I\sqrt{R^2 + (X_L - X_C)^2}$ — (1)

Let Z be the impedence of the circuit
Total P.D across the circuit $V = IZ = OF$ — (2)

$\therefore IZ = I\sqrt{R^2 + (X_L - X_C)^2}$

$Z = \sqrt{R^2 + (X_L - X_C)^2}$ — (3)

from Δ AOF; $\cos\phi = \frac{OA}{OF} = \frac{IR}{IZ} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}$

In this case i.e $X_L > X_C$ from the diagram we find that the total P.D across the circuit leads the current by phase angle 'ϕ'

from Δ AOF; $\tan\phi = \frac{AF}{OA} = \frac{\cancel{I}(X_L - X_C)}{\cancel{I}R}$

$$\tan\phi = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

j- Operator method:

(i) Since the P.D across the resistance $V_R = IR$ is in phase with current; it can be vectorically represented as $\vec{V}_R = \vec{I}R$

(ii) P.D across inductance $V_L = IX_L$ leads the current by phase angle $\pi/2$ and can be vectorically represented as $\vec{V}_L = j\vec{I}X_L$

(iii) P.D across the capacitor $V_C = IX_C$ lags the current, by phase angle $\pi/2$ and can be vectorially represented as $\vec{V}_C = -j\vec{I}X_C$.

∴ The total P.D across the circuit can vectorially be represented as $\vec{V} = \vec{V_R} + \vec{V_L} + \vec{V_C}$

or $\vec{V} = \vec{I}\,[R + j(X_L - X_C)]$ — (1)

Let $z^* = ze^{j\phi}$ be the complex vector impedance of the circuit.

∴ The total P.D across the circuit

$$\vec{V} = \vec{I} z^* = \vec{I} z e^{j\phi} \quad \text{— (2)}$$

Where z = magnitude of the impedance
& ϕ = Phase difference between the P.D & the current.

Equating (1) and (2) :-

$$\vec{I} z [\cos\phi + j\sin\phi] = \vec{I}[R + j(X_L - X_C)]$$

Equating real and imaginary parts :

$$z\cos\phi = R \quad \text{— (3)}$$

$$z\sin\phi = (X_L - X_C) \quad \text{— (4)}$$

∴ Squaring and adding (3) and (4) :

$$z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{— (5)}$$

Dividing eqn (4) by (3) ;

$$\boxed{\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - 1/\omega C}{R}}$$

The P.d across the circuit may lead or lag the current depending on the values of ω, L & C.
from eqn (3), the power factor of the circuit is

$$\boxed{\cos\phi = \frac{R}{z}}$$