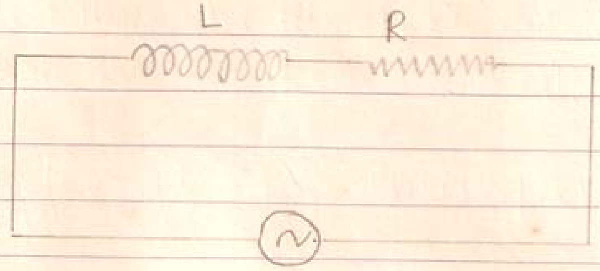
A.C Circuits:

L- R circuit:



L = Self inductance of the coil.
R = Resistance in the circuit. = External resistance + the Ohmic resistance of the coil.

Given: $e = E_p \sin\omega t$ — ① The applied instantaneous e.m.f.
Let i = current flowing in the circuit at the instant t, with I_p as peak value of current.
This circuit can be solved in three different ways:

(1) By writing the e.m.f eqⁿ of the circuit.
(2) By drawing vector diagram of the circuit.
(3) By j-Operator method.

① The e.m.f Eqⁿ method:
The e.m.f Eqⁿ at the instant 't':

$$L\frac{di}{dt} + Ri = e = E_p \sin\omega t \quad \text{— ②}$$

Eqⁿ ② is a 1st Order non-homogenous differential Eqⁿ. Let us assume a trial solution of the eqⁿ

$$i = I_p \sin(\omega t - \phi) \quad — ③$$

where I_p and 'ϕ' are unknown.

$$\therefore \frac{di}{dt} = I_p \omega \cos(\omega t - \phi) \quad — ④$$

Since eqⁿ ③ is a solution of Eqⁿ ②; it must satisfy eqⁿ ②, putting eqⁿˢ ③ & ④ in ② :-

$$I_p [\omega L \cos(\omega t - \phi) + R \sin(\omega t - \phi)] = E_p \sin\omega t$$

$$\text{or } I_p [\omega L \cdot \cos\omega t \cdot \cos\phi + \omega L \cdot \sin\omega t \cdot \sin\phi + R \sin\omega t \cdot \cos\phi - R \cos\omega t \cdot \sin\phi] = E_p \sin\omega t$$

$$\text{or } I_p [\cos\omega t \{\omega L \cos\phi - R \sin\phi\} + \sin\omega t \{\omega L \sin\phi + R \cos\phi\}] = E_p \sin\omega t \quad — ⑤$$

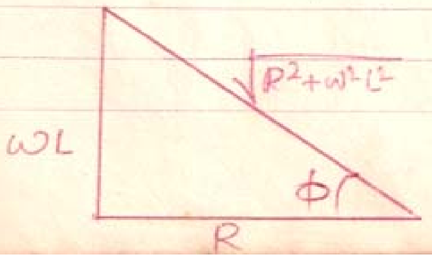
Equating the coefficients of $\cos\omega t$ from both sides of Eqⁿ ⑤ :

$$I_p (\omega L \cos\phi - R \sin\phi) = 0$$

But $I_p \neq 0$ otherwise solution ③ become meaningless.

$$\therefore \omega L \cos\phi - R \sin\phi = 0$$

$$\text{or } \tan\phi = \frac{\omega L}{R} \quad — ⑥$$



$$\therefore \sin\phi = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} \quad — ⑦$$

$$\cos\phi = \frac{R}{\sqrt{R^2 + \omega^2 L^2}} \quad — ⑧$$

Equating the co-efficient of 'Sinωt' from both Sides of Eq^n ⑤ :

$$I_P [\omega L \sin\phi + R\cos\phi] = E_P$$

or $$I_P = \frac{E_P}{R\cos\phi + \omega L \sin\phi}$$

putting eq^{ns} ⑦ and ⑧ :

$$I_P = \frac{E_P}{\frac{R \cdot R}{\sqrt{R^2+\omega^2L^2}} + \frac{\omega L \cdot \omega L}{\sqrt{R^2+\omega^2L^2}}} = \frac{E_P\sqrt{R^2+\omega^2L^2}}{R^2+\omega^2L^2}$$

Or $$I_P = \frac{E_P}{\sqrt{R^2+\omega^2L^2}} \quad \text{—— (9)}$$

Impedence $$Z = \frac{E_P}{I_P} = \sqrt{R^2+\omega^2L^2} \quad \text{—— (10)}$$

putting eq^{ns} ⑥ & ⑨ in ③ :

$$i = \frac{E_P}{\sqrt{R^2+\omega^2L^2}} \sin\left\{\omega t - \tan^{-1}\frac{\omega L}{R}\right\} \quad \text{—— (11)}$$

Comparing eq^n ① and ⑪ we find that the P.d across a L-R circuit leads the current by a phase angle $\phi = \tan^{-1}\frac{\omega L}{R}$.

From eq^n ⑪ we find that the resistence offered to the flow of A.C by the L-R circuit i.e the Impedance of the circuit

$$Z = \frac{E_P}{I_P} = \sqrt{R^2 + X_L^2} \quad \text{or} \quad Z = \sqrt{R^2+\omega^2L^2}$$

from eq^n ⑧, power factor of the L-R circuit

$$\cos\phi = \frac{R}{\sqrt{R^2+\omega^2L^2}}$$

(ii) We now solve the above circuit by vector diagram method:

Although alternating current and p.d are not vector quantities but when we consider the instantaneous values of current and p.d simultaneously in a component since they may have a phase difference between them, this phase difference may be diagramitically be indicated by angle between two lines, the lines represent the current and p.d vectorially & the length of the lines are proportional to the value of current and p.d.

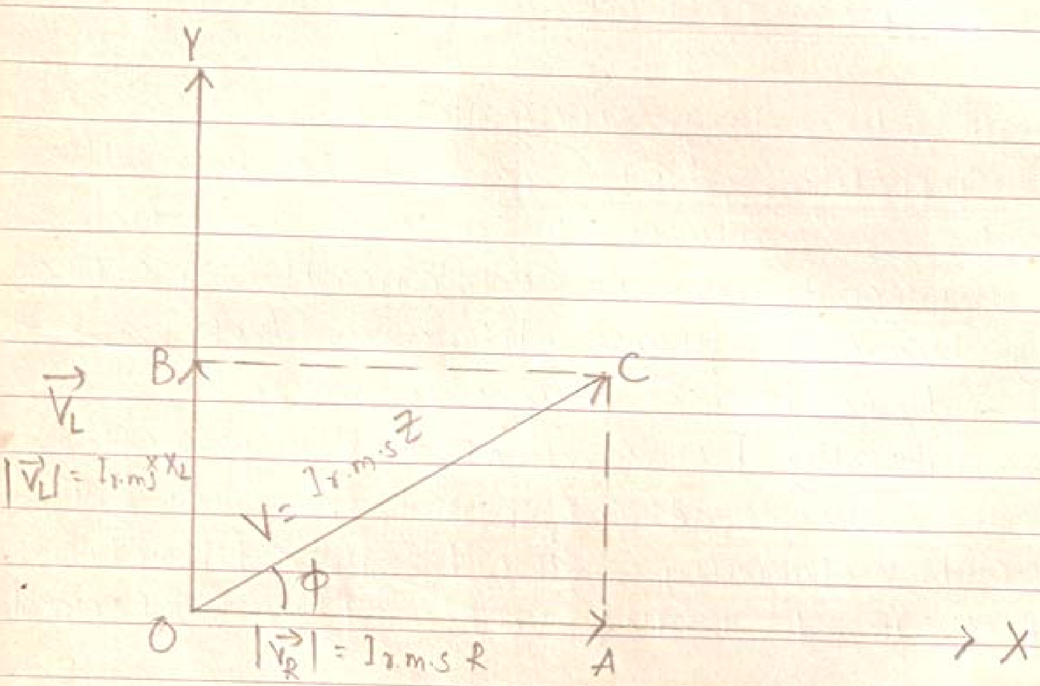
While drawing the vector diagram the following points are to be followed:

(a) In a series circuit, since the instantaneous current is same in all the components, It is represented along X-axis.

(b). The p.d across a resistance is in phase with current & hence can be represented by a vector in the same direction as the current vector.

(c). The p.d across an inductance leads the current by a phase angle $\pi/2$ and hence can be represented by a vector along Y-axis

(d). The P.d across a capacitor lags the current by a phase angle $\pi/2$ and hence can be represented by a vector along − Y axis.



Let x-axis represent the direction of current vector.

(i) The P.d across the resistance $V_R = I_{r.ms} \cdot R$ and since this P.d is in phase with current it can be represented by a vector $\vec{OA}$ of length $V_R = I_{r.ms} R$ along x-axis $\vec{OA} = \vec{V_R}$.

(ii) The P.d across the inductance $V_L = I_{r.ms} \times X_L$ and since this P.d leads the current by phase angle $\pi/2$ it can be represented by a vector $\vec{OB}$ of length $OB = V_L = I_{r.ms} X_L$ along Y ax

The resultant P.d V across the circuit can be found by law of parallelogram of vectors. The diagonal $\vec{OC}$ represents the resultant P.d across the circuit both in length and phase.

The magnitude of the resultant P.d across the circuit $|\vec{V}| = V = \sqrt{|\vec{V_R}|^2 + |\vec{V_L}|^2 + 2|\vec{V_R}||\vec{V_L}|\cos 90}$

$$\text{or } V = \sqrt{V_R^2 + V_L^2} = \sqrt{I_{r.m.s}^2 R^2 + I_{r.m.s}^2 X_L^2}$$

$$V = I_{r.m.s}\sqrt{R^2 + X_L^2} \quad \text{— (1)}$$

Let Z be the impedance of the circuit

$$\therefore V = I_{r.m.s} Z \quad \text{— (2)}$$

Equating (1) and (2):

$$\cancel{I_{r.m.s}} Z = \cancel{I_{r.m.s}} \sqrt{R^2 + X_L^2}$$

$$\text{or } Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2} \quad \text{— (3)}$$

Let the resultant P.d along $\vec{OC}$ leads the current by phase angle 'ϕ' i.e $A\hat{O}C = \phi$

$$\text{from } \Delta AOC, \quad \tan\phi = \frac{AC}{OA} = \frac{OB}{OA} = \frac{\cancel{I_{r.m.s}} X_L}{\cancel{I_{r.m.s}} R}$$

$$\tan\phi = \frac{X_L}{R} \quad \text{or } \tan\phi = \frac{\omega L}{R} \quad \text{— (4)}$$

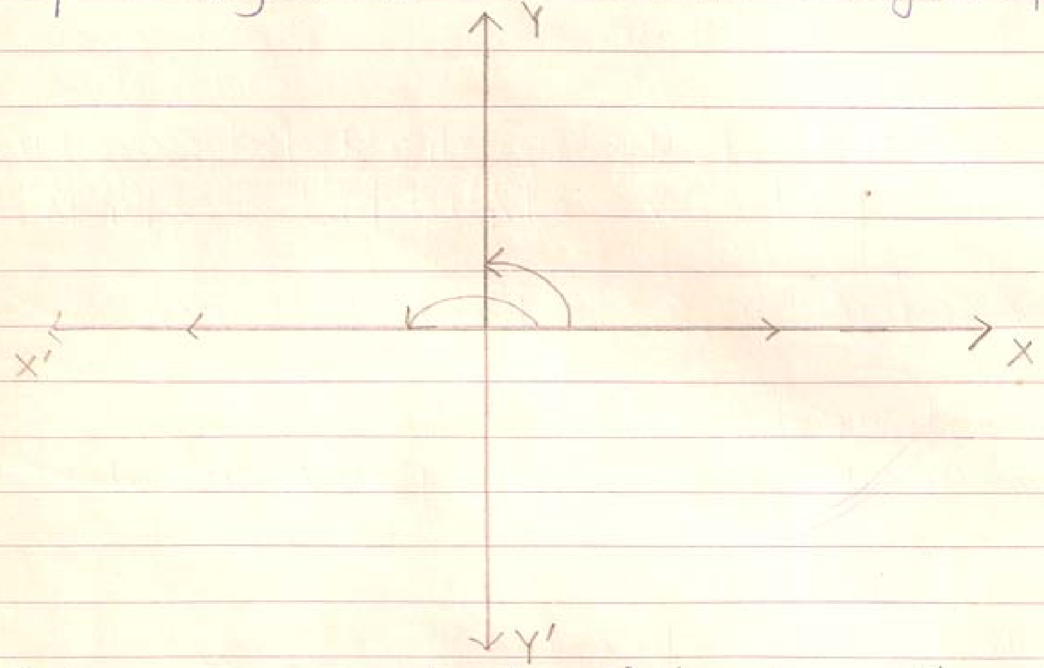
The power factor of the circuit is $\cos\phi$

$$\text{from } \Delta AOC, \quad \cos\phi = \frac{OA}{OC} = \frac{\cancel{I_{r.m.s}} R}{\cancel{I_{r.m.s}} Z} = \frac{R}{Z}$$

$$\text{or } \boxed{\cos\phi = \frac{R}{\sqrt{R^2 + \omega^2 L^2}} \quad \text{— (5)}}$$

3rd method : j - operator method :

$\vec{x}$ & $-\vec{x}$ are two different vectors. They have equal magnitude and direction are just opposite.



$\vec{x}$ is along x-axis & $-\vec{x}$ is along X' axis.

$$-\vec{x} = (-1)\vec{x} = j^2\vec{x} \quad — (1)$$

where $j = \sqrt{-1}$

Thus from (1) we find that multiplication of a vector by j^2 keeps the magnitude of the vector unchanged but simply rotates the vector by 180°.

Eq^n (1) can be written as :

$$-\vec{x} = j(j\vec{x}) = j\vec{y} \quad — (2)$$

where $\vec{y} = j\vec{x} \quad — (3)$

n eq^n (2) and (3) we can conclude that the multiplication of a vector by j keeps the magnitude of the vector unchanged but simply rotates the vector by $\pi/2$.

L-R circuit by j-operator method:

(i) Let $\vec{I}_{r.m.s}$ represent the current in the series of L-R circuit having magnitude $I_{r.m.s}$.

(ii) Since the p.d across the resistance V_R is in phase with the current $I_{r.m.s}$ hence $\vec{V_R}$ & $\vec{I}_{r.m.s}$ both will have same directions.

Since $V_R = I_{r.m.s} R$ hence $\vec{V_R} = \vec{I}_{r.m.s} R$.

(iii) Since the p.d across the inductance V_L leads the current I by phase angle $\pi/2$ hence direction of $\vec{V_L}$ will be rotated by 90° along anticlockwise direction w.r.t the direction of vector $\vec{I}_{r.m.s}$.

Since $V_L = X_L I_{r.m.s}$ $\quad V_L = I_{r.m.s} X_L$

In order to rotate the direction of $\vec{I}_{r.m.s}$ by 90° anticlockwise we have to multiply $\vec{I}_{r.m.s}$ by j & then we get the direction of $\vec{V_L}$.

$$\therefore \vec{V_L} = j\,\vec{I}_{r.m.s} X_L.$$

Let $\vec{V}$ represent the total p.d across the

From law of vector addition:

$$\vec{V} = \vec{V_R} + \vec{V_L}$$

$$\text{or } \vec{V} = \vec{I}_{r.m.s} R + j\,\vec{I}_{r.m.s} X_L$$

$$\vec{V} = \vec{I}_{r.m.s}\left[R + jX_L\right] \quad \text{--- (1)}$$

From eqⁿ ① we find that $\vec{V}$ is complex.
Let Z^* represent the complex impedence of the circuit i.e $\vec{V} = \vec{I}_{r.m.s} Z^*$

$$\vec{V} = \vec{I}_{r.m.s} Z e^{j\phi} \quad \text{— (2)}$$

where $Z^* = Z e^{j\phi}$

where Z represents the value of the impedance & 'ϕ' represents the phase difference between the $\vec{V}$ & $\vec{I}_{r.m.s}$ i.e phase difference between p.d across the circuit & the current i.e angle between the vectors $\vec{V}$ & $\vec{I}_{r.m.s}$.

Equating ① and ②:

$$\vec{V} = \vec{I}_{r.m.s} Z^* = \cancel{I_{r.m.s}} Z e^{j\phi} = \cancel{I_{r.m.s}} [R + jX_L]$$

$$\text{or} \quad Z e^{j\phi} = R + jX_L$$

$$Z[\cos\phi + j\sin\phi] = R + jX_L$$

Equating the real and imaginary parts from both sides:

$$Z\cos\phi = R \quad \text{— (4)}$$

$$Z\sin\phi = X_L \quad \text{— (5)}$$

Squaring and adding eqⁿ ④ and ⑤:

$$Z^2 = R^2 + X_L^2$$

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2} \quad \text{— (6)}$$

Dividing eqⁿ ⑤ by ④:

$$\frac{Z\sin\phi}{Z\cos\phi} = \frac{X_L}{R}$$

$$\boxed{\tan\phi = \frac{X_L}{R} = \frac{\omega L}{R}} \quad \text{— (7)}$$

from ④; $\cos\phi = \frac{R}{Z}$ putting ⑥:–

$$\boxed{\cos\phi = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}} \quad \text{— (8)}$$

Equation (6) gives the impedance of the L-R circuit. Equation (7) gives the phase angle ϕ by which the P.d across the circuit leads the current.
Equation (8) gives the power factor of the circuit.