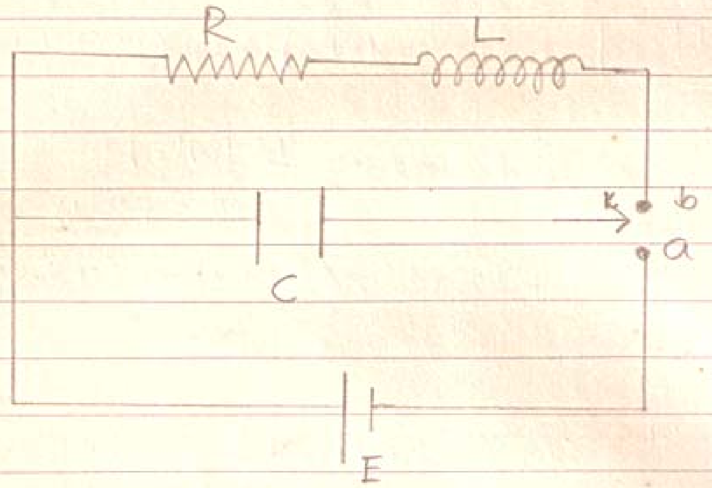
The discharge of a Capacitor, through a resistance and inductance in Series:



The tapping key K is first connected to 'a', the Capacitor get charged fully.
Then the key K is connected to b, the fully charged Capacitor get discharged through the inductance and resistance in Series.

Given:

L = Self inductance of the coil.
C = The capacitance of the Capacitor.
R = The resistance in the circuit.
Q_0 = The initial Charge Stored in the Capacitor.

Let Q be the charge Stored in the Capacitor at any instant 't' during discharge.
Let i be the current flowing in the Circuit at that instant of time t.
The e.m.f eq^n of the circuit at the instant 't'
(Since the external e.m.f E is zero)

$$L\frac{di}{dt} + iR + \frac{Q}{C} = 0 \quad \text{But } i = \frac{dQ}{dt} \therefore \frac{di}{dt} = \frac{d^2Q}{dt^2}$$

$\therefore L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$

or $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$ — (1)

put $\frac{R}{L} = 2b = \text{Const.}$ & $\frac{1}{LC} = k^2 = \text{Const.}$ — (2)

putting eq^n (2) in (1):

$$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + k^2Q = 0 \quad \text{— (3)}$$

Eq^n (3) is a Second Order homogenous differential Eq^n & can be solved by D-Operator method.

put $\frac{d}{dt} = D$ & $\frac{d^2}{dt^2} = D^2$

In terms of D-operator eq^n (3) becomes

$$D^2(Q) + 2b\,D(Q) + k^2(Q) = 0$$

or $[D^2 + 2bD + k^2](Q) = 0$ — (4)

The bracketed term of eq^n (4) being quadratic it has two roots. Let m_1 and m_2 be the two roots.

$m_1 = -b + \sqrt{b^2-k^2} = -b+m$ where $m = \sqrt{b^2-k^2}$

$m_2 = -b - \sqrt{b^2-k^2} = -b-m$

Also $(m_1+m_2) = -2b$ & $m_1 m_2 = k^2$ — (5)

Putting eq^n (5) in (4):-

$$[D^2 - (m_1+m_2)D + m_1m_2](Q) = 0$$

$$[D(D-m_1) - m_2(D-m_1)](Q) = 0$$

or $(D-m_1)(D-m_2)(Q) = 0$

Either $(D-m_1)(Q)=0$ or $(D-m_2)(Q)=0$

or $D(Q)=m_1 Q$ or $\frac{dQ}{dt}=m_1 Q$ or $\frac{dQ}{Q}=m_1 dt$

Integrating with sides:

$$\log_e Q = m_1 t + \log A$$

$$Q = A e^{m_1 t}$$

Similarly proceeding with the other solution,

$Q = B e^{m_2 t}$ where A and B are unknown constants of integration.

The general solution of eq^n ① is the sum of these two solutions $Q = A e^{m_1 t} + B e^{m_2 t}$

putting the values of m_1 & m_2

$$Q = A e^{(-b+m)t} + B e^{(-b-m)t}$$

$$Q = e^{-bt}\left[A e^{mt} + B e^{-mt}\right] \quad \text{—⑥}$$

eq^n ⑥ gives the charge stored in the capacitor at any instant 't' during discharging.

Differentiating eq^n ⑥ w.r.to 't' we get the current flowing through the circuit at any instant t

$$i = \frac{dQ}{dt} = -b e^{-bt}\left[A e^{mt} + B e^{-mt}\right] + e^{-bt}\left[mA e^{mt} - m\right.$$

—⑦

We now evaluate the consts A and B, applying the initial condition.

(i) at $t=0$; $Q=Q_0$ from ⑥

$$Q_0 = 1\left[A\cdot 1 + B\cdot 1\right] \quad \therefore A+B=Q_0 \quad \text{—⑧}$$

(ii) at $t=0$; $i=0$ from ⑦ :

$$0 = -b.1[A+B] + 1.m[A-B]$$

$$\therefore A-B = \frac{b}{m}(A+B) \quad \text{Putting eq}^n \text{ ⑧}$$

$$A-B = \frac{b}{m}Q_0 \quad \text{— ⑨}$$

Adding eq^n ⑧ and ⑨ we get $A = \frac{Q_0}{2}\left(1+\frac{b}{m}\right)$

Substracting eq^n ⑨ from ⑧ we get $B = \frac{Q_0}{2}\left(1-\frac{b}{m}\right)$

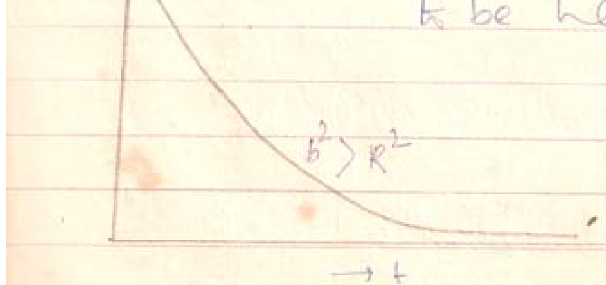
Thus knowing the unknown constants A and B Charge Q & the current 'i' in the circuit at any instant 't' can be calculated, using eq^n ⑥ and ⑦ respectively.

We now discuss few special cases:

Case I: Let $\frac{R^2}{4L^2} > \frac{1}{LC}$ i.e $b^2 > k^2$ In this case, $m = \sqrt{b^2-k^2}$ is real & $m < b$

$\therefore m_1 = -b+m = (-)ve$; $m_2 = -b-m = -ve$.

Hence in this case m_1 and m_2 both being negative Q falls to zero in an exponential fashion.



The discharge in this case being very slow, it is said to be heavily damped.

Exponential decay of charge

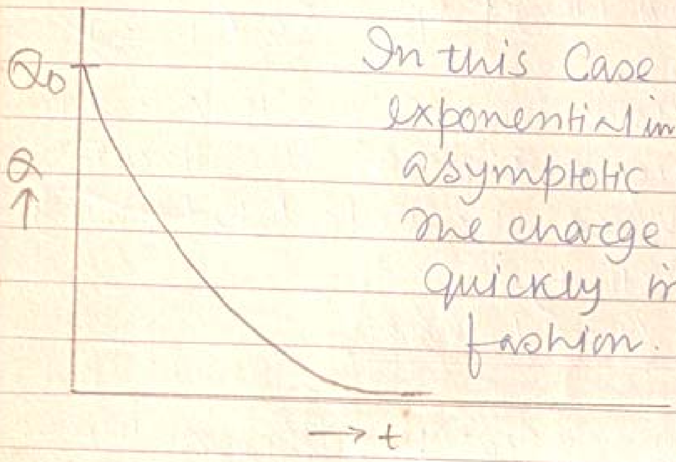
Case II : Let $\frac{R^2}{4L^2} = \frac{1}{LC}$ or $b^2 = k^2$ $m = 0$

$$\therefore Q = e^{-bt}\left[A e^{0.t} + B e^{-0.t}\right]$$

$$\text{or } Q = e^{-bt}\left[A + B\right]$$

putting $A + B = Q_0$ from (8) :

$$Q = Q_0 e^{-bt} \quad \text{—(11)}$$



In this case the discharge is exponential in fashion and is asymptotic to the time axis. The charge falls to zero very quickly in an exponential fashion. The discharge is said to be critically damped.

Case III : $\frac{R^2}{4L^2} < \frac{1}{LC}$ or $b^2 < k^2$ $\therefore m = \sqrt{b^2 - k^2}$ = Imaginary.

put $m = \sqrt{-1(k^2 - b^2)} = j\omega$

where $\omega = \sqrt{k^2 - b^2}$ is real

$$\therefore Q = e^{-bt}\left[A e^{j\omega t} + B e^{-j\omega t}\right]$$

$$\text{Or } Q = e^{-bt}\left[A(\cos\omega t + j\sin\omega t) + B(\cos\omega t - j\sin\omega t)\right]$$

$$\text{Or, } Q = e^{-bt}\left[(A+B)\cos\omega t + j(A-B)\sin\omega t\right]$$

putting eqns (8) and (9)

$$Q = e^{-bt}\left[Q_0\cos\omega t + j\frac{Q_0 b}{m}\sin\omega t\right]$$

Or $Q = Q_0 e^{-bt}\left[\cos\omega t + j\dfrac{b}{j\omega}\sin\omega t\right]$

or, $Q = \dfrac{Q_0 e^{-bt}}{\omega}\left[\omega\cos\omega t + b\sin\omega t\right]$

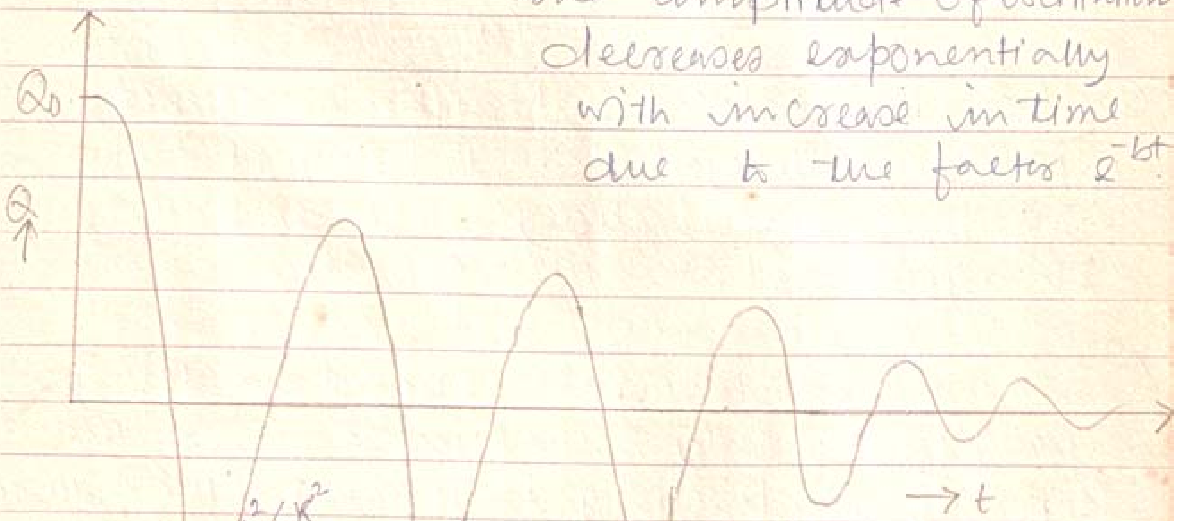
put $\omega = K\sin\alpha$ & $b = K\cos\alpha$ where α is unknown

$\therefore Q = \dfrac{Q_0 e^{-bt}}{\omega}\left[K\sin\alpha\cdot\cos\omega t + K\cos\alpha\cdot\sin\omega t\right]$

$$Q = \frac{Q_0 K e^{-bt}}{\omega}\left[\sin(\omega t + \alpha)\right] \quad \text{—— (12)}$$

From eq^n (12) we find that the discharge is also oscillatory in fashion.

The amplitude of oscillation decreases exponentially with increase in time due to the factor e^{-bt}.



Oscillating decay of charge.

The frequency of oscillation is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{K^2 - b^2} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \text{—— (13)}}$$

The logarithmic decrement will be $b = \dfrac{R}{2L}$