Damped vibration :- When the vibration of a particle is opposed by some force; frictional in nature, such as the resistive force due to air or the medium in which the particle vibrates; the motion is said to be damped vibration.

Given: m = mass of the particle.

x = The displacement of the particle at an instant of time t.

For a particle executing damped vibration the forces acting on the particle are :-

(i) The restoring force proportional to the displacement with a negative sign.

$\boxed{R.F = -ax}$ where a = Constant of proportionality known as restoring force per unit displacement.

(ii) The damping force; proportional to the velocity with a negative sign.

$\boxed{D.F = -b\cdot\frac{dx}{dt}}$ Where b = Constant of proportionality; known as damping force per unit ~~[illegible]~~ velocity.

Using Newtons equation of motion;

$$m\times\frac{d^2x}{dt^2} = \text{Resultant force} = -ax - b\frac{dx}{dt}$$

$$\text{or, } \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{a}{m}x = 0$$

Put $\frac{a}{m} = \mu^2$ = Restoring force per unit displacement per unit mass.

$\frac{b}{m} = 2k$ = The damping force per unit velocity per unit mass.

$$\therefore \boxed{\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \mu^2 x = 0} \quad \text{——— (1)}$$

Equation (1) is a second order homogenous differential equation which represents the damped vibration. We now solve equation (1) by D-Operator method

Put $\frac{d}{dt} = D$, $\frac{d^2}{dt^2} = D^2$

Using D-operators, equation (1) becomes

$$D^2(x) + 2kD(x) + \mu^2(x) = 0$$

$$\text{or} \quad [D^2 + 2kD + \mu^2](x) = 0 \quad \text{——— (2)}$$

The bracketed term in equation (2) is quadratic & hence has two roots. Let m_1 and m_2 be the two roots of that quadratic expression.

$$\therefore m_1 = \frac{2k + \sqrt{4k^2 - 4.1.\mu^2}}{2.1} = -k + \sqrt{k^2 - \mu^2}$$

$$= -k + m$$

Where $m = \sqrt{k^2 - \mu^2}$

$$m_2 = -k - m$$

$$(m_1 + m_2) = -2k \text{ \& } m_1 m_2 = \mu^2 \quad \text{——— (3)}$$

Putting equation (3) in (2)

$$[D^2 - (m_1 + m_2)D + m_1 m_2](x) = 0$$

$$\text{or} \quad [D(D - m_1) - m_2(D - m_1)](x) = 0$$

$$\text{or,} \quad (D - m_1)(D - m_2)(x) = 0$$

Either $(D - m_1)(x) = 0$ or $(D - m_2)(x) = 0$

or, $D(x) = m_1 x$ or $\frac{dx}{dt} = m_1 x$ or $\boxed{\frac{dx}{x} = m_1 dt}$

Integrating both sides; $\log x = m_1 t + \log A$

or $\log \frac{x}{A} = m_1 t$ or Taking anti-log;

$$\boxed{x = Ae^{m_1 t}}$$

Similarly, proceeding with ~~otle~~ other solution $(D-m_2)(x)$

$$\boxed{x = Be^{m_2 t}}$$

The resultant solution of the homogenous differential equation (1) is the sum of these two solutions

$$\therefore x = Ae^{m_1 t} + Be^{m_2 t} \text{ —— } (3)$$

where A & B are two unknown constants of integration.

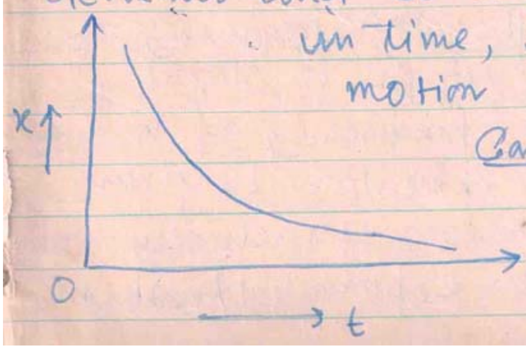
Discussions: Case I: let the damping be very large hence $k \gg \mu$

$$m = \sqrt{k^2 - \mu^2} \quad \text{or} \quad m^2 = k^2 - \mu^2 \quad \therefore m^2 < k^2$$

or $m < k \quad \therefore m_1 = -k + m = -ve$

$m_2 = -k - m = -ve$

Since in this case, m_1 & m_2 both are negative hence with increase in time both the terms decreases and so x decreases rapidly with increase in time, in an exponential fashion. The motion is said to be heavily damped.



Case II: let the damping be there but be very small $\boxed{k < \mu}$ $\therefore m = \sqrt{k^2 - \mu^2}$ = imaginary

Or, $m = \sqrt{-1(\mu^2 - k^2)} = j\sqrt{\mu^2 - k^2}$ or $\boxed{m = j\gamma}$

where $j = \sqrt{-1}$ & $\gamma = \sqrt{\mu^2 - k^2}$ = real

Equation (3) can be written as;

$$x = Ae^{(-k+m)t} + Be^{(-k-m)t}$$

$$= e^{-kt}\left[Ae^{mt} + Be^{-mt}\right]$$

Putting $\underline{m = j\gamma}$

$$x = e^{-kt}\left[Ae^{j\gamma t} + Be^{-j\gamma t}\right]$$

$$\text{or } x = e^{-kt}\left[A\{\cos\gamma t + j\sin\gamma t\} + B\{\cos\gamma t - j\sin\gamma t\}\right]$$

$$\text{or, } x = e^{-kt}\left[\cos\gamma t\{A+B\} + j(A-B)\sin\gamma t\right] \text{ --- (4)}$$

Put $A + B = c\sin\delta$ —— (5)

$j(A-B) = c\cos\delta$ —— (6)

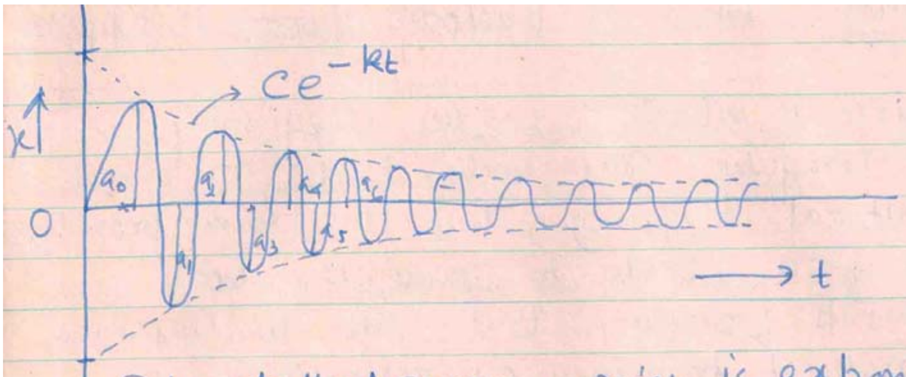
where c & δ are two unknown constants.

Putting equation (5) & (6) in (4):

$$x = e^{-kt}\left[c\sin\delta.\cos\gamma t + c\cos\delta\sin\gamma t\right]$$

$$\text{or, } \boxed{x = ce^{-kt}.\sin(\gamma t + \delta)} \text{ —— (7)}$$

Equation (7) represents the damped vibration when the damping is very small. The frequency of damped vibration is $\gamma = \sqrt{\mu^2 - k^2}$ is slightly less than 'μ' the natural frequency of vibration. The amplitude of damped vibration is given by ce^{-kt} which decreases gradually with the increase in time; in an exponential fashion.



The dotted curve, which is exponential curve represents the curve between the amplitude & time

let $a_0, a_1, a_2, a_3, a_4, a_5$ ..... represent the amplitude ; at instants $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \frac{7T}{4}, \frac{9T}{4}$ ..... respectively.

$\therefore a_0 = ce^{-kT/4}$; $a_1 = ce^{-k3T/4}$; $a_2 = ce^{-k5T/4}$

$a_3 = ce^{-k7T/4}$, $a_4 = ce^{-k9T/4}$ ....

$$\therefore \frac{a_0}{a_1} = \frac{\cancel{c}e^{-kT/4}}{\cancel{c}e^{-k3T/4}} = e^{k(3-1)T/4} = e^{kT/2}$$

$$\frac{a_1}{a_2} = \frac{\cancel{c}e^{-k3T/4}}{\cancel{c}e^{-k5T/4}} = e^{kT/2}$$

$$\therefore \frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \frac{a_3}{a_4} = \ldots\ldots = e^{kT/2} = e^{\lambda}$$

Thus we find that the ratio of successive amplitudes is constant ; equal to e^{λ} where $\lambda = kT/2$ = constant, known as logarithmic decreament.