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ELECTROSTATICS :

Gauss's theorem : " Total flux through a closed surface of any size and any shape is equal to $\frac{1}{\epsilon}$ times the charge enclosed by that closed surface. where ϵ is the permitivity of the medium surrounding the charge"

$$d\phi = \vec{E} \cdot \vec{dA}$$

$$\phi = \oiint_A d\phi = \oiint_A \vec{E} \cdot \vec{dA} = \frac{Q}{\epsilon}$$

Let ρ be the density of charge at a point enclosed by the surface.
Consider an element of volume '$d\tau$' surrounding that point.
Charge enclosed by that element $= \rho\, d\tau$
Total charge enclosed by the surface $= \iiint_\tau \rho\, d\tau$ — (2)

$$\oiint_A \vec{E} \cdot \vec{dA} = \frac{1}{\epsilon} \iiint_\tau \rho\, d\tau \quad — (3)$$

Converting the L.H.S of eq^n (3) from surface integral to volume integral by applying Gauss's theorem

$$\oiint_A \vec{E} \cdot \vec{dA} = \iiint_\tau (\nabla \cdot \vec{E})\, d\tau \quad — (4)$$

putting eq^n (4) in (3) :

$$\iiint_\tau (\nabla \cdot \vec{E})\, d\tau = \frac{1}{\epsilon} \iiint \rho\, d\tau$$

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$$\therefore \boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon}} \quad \text{—(5)}$$

$$\vec{E} = -\nabla V \quad \therefore \boxed{\nabla \cdot \vec{E} = -\nabla \cdot \nabla V = -\nabla^2 V} \quad \text{—(6)}$$

$$\therefore \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \quad \text{—(7)}$$ Poisson's equation.

If there is no charge $\rho = 0$

$$\boxed{\nabla^2 V = 0} \quad \text{—(8)}$$ Laplace's equation.

In rectangular Cartesian co-ordinates Laplaces equation is

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

The potential 'V' is a Scalar point function which varies from point to point in Space : $V = V(x, y, z)$

$$V = X(x)\, Y(y)\, Z(z)$$

where X is a function of x- co-ordinate only and remains constant with change in the other two co-ordinates.

Similarly Y and Z are also some unknown functions of y and z co-ordinates only.

$$\therefore \frac{\partial V}{\partial x} = \frac{\partial (X_x)}{\partial x} Y(y)\, Z(z)$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} Y(y)\, Z(z)$$

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$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 Y}{\partial y^2} X(x) Z(z)$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 Z}{\partial z^2} X(x) Y(y)$$

$$\nabla^2 V = \frac{\partial^2 X}{\partial x^2} Y(y) Z(z) + \frac{\partial^2 Y}{\partial y^2} X(x) Z(z) + \frac{\partial^2 Z}{\partial z^2} X(x) Y(y) = 0$$

Dividing throughout by $X(x) Y(y) Z(z)$

$$\nabla^2 V = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

Putting $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\alpha^2$

$$\frac{\partial^2 X}{\partial x^2} + \alpha^2 X = 0$$

put $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$

$$D^2(X) + \alpha^2(X) = 0$$

or $[D^2 + \alpha^2](X) = 0$

or $[D + i\alpha][D - i\alpha](X) = 0$

Either $(D - i\alpha)X = 0$ or $(D + i\alpha)X = 0$

$$D(X) = i\alpha X$$

or $\frac{dX}{X} = i\alpha dx$ or $\log X = i\alpha x + \log A_1$

$\therefore X = A_1 e^{i\alpha x}$

$X = A_2 e^{-i\alpha x}$

$X = A_1 e^{i\alpha x} + A_2 e^{-i\alpha x}$

$Y(y) = B_1 e^{i\beta y} + B_2 e^{-i\beta y}$

$Z(z) = C_1 e^{i\gamma z} + C_2 e^{-i\gamma z}$

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$$\therefore V(x,y,z) = \sum_{r,s,t} \left(A_1^r e^{i\alpha_r x} + A_2^r e^{-i\alpha_r x} \right) \left(B_1^s e^{i\beta_s y} + B_2^s e^{i\beta_s y} \right) \cdot \left(C_1^t e^{i\gamma_t z} + C_2^t e^{-i\gamma_t z} \right)$$