Laplace’s equation in spherical polar co-ordinate

SelfStudy.in

Laplace's equation in Spherical polar co-ordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad \text{—(1)}$$

By using the method of Separation of variables ; put $V = RS$ in eq^n(1)

$R = R(r)$ = Radial function
$S = S(\theta, \phi)$ Spherical harmonics.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\cdot S\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, R \cdot \frac{\partial S}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} R \cdot \frac{\partial^2 S}{\partial \phi^2} = 0$$

Dividing throughout by 'RS/r^2'

$$\frac{1}{R \cancel{r^2}}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{S \cancel{r^2} \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S}{\partial \theta}\right) + \frac{1}{S \cancel{r^2} \sin^2\theta}\frac{\partial^2 S}{\partial \phi^2} = 0 \quad \text{—(2)}$$

In eq^n(2) we find that, the 1st term is a function of 'r' only, while the remaining two terms are independent of r. Hence this equation is satisfied if we take

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = K = \text{const} \quad \text{— (3)}$$

SelfStudy.in

$$\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right)+\frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2}=-K=\text{const} \quad (4)$$

where K is constant. The solutions of these equations take a simpler form if we take the constant K in the form of $l(l+1)$

The solution of eq^n (3) is found to be

$$R = A r^l + \frac{B}{r^{l+1}} \quad \text{——} (5)$$

Solution:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = K = \text{constant}$$

$$\text{or } r^2\frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} - KR = 0$$

putting $K = l(l+1)$

$$\therefore \frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} - \frac{l(l+1)R}{r^2} = 0 \quad \text{——} (x)$$

Standard form: $\frac{\partial^2 R}{\partial r^2} + p(r)\frac{\partial R}{\partial r} + Q(r)R = 0$

where $p(r) = \frac{2}{r}$, $Q(r) = -\frac{l(l+1)}{r^2}$

If $m(m-1)\cancel{x} + mPx + Qx^2 = 0$ then $y = x^m$ will be a solution of the equation.

$$l(l-1) + lPr + Qr^2$$

$$= l(l-1) + l\cdot\frac{2}{\cancel{r}}\cdot\cancel{r} + \frac{(-K)}{\cancel{r^2}}\cdot\cancel{r^2} \quad \Big| \text{ putting } K = l(l+1)$$

$$= \cancel{l^2} - \cancel{l} + 2l - \cancel{l^2} - \cancel{l} = 0$$

$\therefore r^l$ will be a solution

Put $R = v r^l$ where v is also a function of r.

$$\frac{dR}{dr} = \frac{dv}{dr} \cdot r^l + l(r^{l-1}) \cdot v$$

$$\text{or } \frac{d^2R}{dr^2} = \frac{d^2v}{dr^2} r^l + \frac{dv}{dr} l r^{(l-1)} + \frac{dv}{dr} l r^{(l-1)} + v l(l-1) r^{l-2}$$

$$\text{or } \frac{d^2R}{dr^2} = \frac{d^2v}{dr^2} r^l + 2\frac{dv}{dr} l r^{(l-1)} + v l(l-1) r^{l-2}$$

putting the values of $\frac{d^2R}{dr^2}$ and $\frac{dR}{dr}$ in the given equation in ⓧ :-

$$r^l\left[\frac{d^2v}{dr^2} + \frac{2l}{r}\cdot\frac{dv}{dr} + \frac{v\,l(l-1)}{r^2}\right]$$

$$+ \frac{2}{r} r^l\left[\frac{dv}{dr} + \frac{v \cdot l}{r}\right] - \frac{l(l+1)}{r^2} v r^l = 0$$

$$\text{or } \frac{d^2v}{dr^2} + \left[\frac{2l}{r} + \frac{2}{r}\right]\frac{dv}{dr} + \left[\cancel{l^2} - \cancel{l} + \cancel{2l} - \cancel{l^2} - \cancel{l}\right]\frac{v}{r^2} = 0$$

$$\text{or } \frac{d^2v}{dr^2} + \frac{2}{r}(l+1)\frac{dv}{dr} = 0$$

put $\frac{dv}{dr} = y$, $\frac{d^2v}{dr^2} = \frac{dy}{dr}$

$$\frac{dy}{dr} + \frac{2(l+1)}{r} y = 0$$

$$\text{or } \frac{1}{2}\frac{dy}{y} = -(l+1)\frac{dr}{r} \quad \Big| \text{ Integrating both sides.}$$

$$\frac{1}{2}\log y = -(l+1)\log r + \log A$$

$\therefore \log \frac{y^{1/2}}{A} = -(l+1)\log r$

or $\log \frac{y^{1/2}}{A} = \log r^{-(l+1)}$

or $y^{1/2} = A\, r^{-(l+1)}$

or $y = A^2\, r^{-2(l+1)}$

or $y = B\, r^{-2(l+1)}$

$\frac{dv}{dr} = B\, r^{-2(l+1)}$ or $dv = B\, r^{-2(l+1)}\, dr$

Integrating both sides

$v = B\frac{r^{-2l-2+1}}{-2l-1} + C$

or $v = \frac{B}{-(2l+1)} r^{-2l-1} + C$

$\therefore R = v r^{l} = E\, r^{-2l-1+l} + C r^{l}$

$R = C r^{l} + E\, r^{-(l+1)}$

or $\boxed{R = C r^{l} + \frac{E}{r^{(l+1)}}}$

Spherical harmonics:

$$\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} = -l(l+1) \quad (6)$$

SelfStudy.in

Put $S = P(\theta)Q(\phi)$ and multiplying by $S\sin^2\theta$

$$\therefore \sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P}{\partial\theta} Q\right) + P\frac{\partial^2 Q}{\partial\phi^2} + l(l+1)S\sin^2\theta = 0$$

Dividing throughout by $P(\theta)Q(\phi)$

$$\frac{\sin\theta}{P}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} + l(l+1)\sin^2\theta = 0$$

or $$\left[\frac{\sin\theta}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + l(l+1)\sin^2\theta\right] + \frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} = 0 \quad \text{—(7)}$$

Since the first bracketed term is a function of 'θ' only and the second term is a function of 'ϕ' only the eq^n (7) is satisfied only if

$$\frac{\sin\theta}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + l(l+1)\sin^2\theta = -\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} = m^2 \quad \text{—(8)}$$

Where 'm' is a constant.

from (8):

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} = -m^2 \quad \text{—(9)}$$

or $\dfrac{\partial^2 Q}{\partial\phi^2} + m^2 Q = 0$

or $[D^2 + m^2]Q = 0$ $\qquad\Big|\ D = \dfrac{d}{d\phi}$

or $(D - im)(D + im)Q = 0$

Either $(D - im)Q = 0$ or $(D + im)Q = 0$

SelfStudy.in

$\therefore \frac{dQ}{d\phi} = imQ$ or $\frac{dQ}{Q} = im\,d\phi$

Integrating both sides: $\log_e Q = im\phi + \log c_1$

$\therefore \frac{Q}{c_1} = e^{im\phi}$ or $Q = c_1 e^{im\phi}$

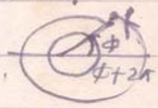
Similarly the other solution is ; $Q = c_2 e^{-im\phi}$

General solution of the azimuthal Eq^n (9)

$$Q(\phi) = ce^{\pm im\phi} \quad \text{— (10)} \qquad \phi^*(\phi) = ce^{\mp im\phi}$$

In order that the potential be single valued

$$e^{\pm im\phi} = e^{\pm im(\phi+2\pi)} \quad \text{— (11)}$$ because 'ϕ' and '$\phi+2\pi$' represent the same point



$$\cos m\phi \pm i\sin m\phi = \cos m(2\pi+\phi) \pm i\sin m(2\pi+\phi)$$

$\cos(2m\pi + m\phi) = \cos m\phi$ only when 'm' is an integer

Eq^n (11) is satisfied only if 'm' is an integer.

The function Q_m (the value of Q when const is m) are normalised by choosing suitable constant c.

$$\int_0^{2\pi} Q_m^* \, Q_m \, d\phi = 1$$

or, $$c^2 \int_0^{2\pi} e^{\mp im\phi} \cdot e^{\pm im\phi} \cdot d\phi = 1.$$

SelfStudy.in

Or, $c^2 \int_0^{2\pi} d\phi = 1$ or $c^2 \cdot 2\pi = 1$ or $\boxed{c = \frac{1}{\sqrt{2\pi}}}$

$$\therefore \; Q_m = \frac{1}{\sqrt{2\pi}} e^{\pm im\phi} \quad \text{——} \quad (13)$$

The function Q_m are orthogonal.

$$\int_0^{2\pi} Q_m^* \cdot Q_n \, d\phi = 0 \quad \text{when } m \neq n$$
$$= 1 \quad \text{when } m = n$$

i.e $\int_0^{2\pi} Q_m^* Q_n \, d\phi = \delta_{mn}$

δ_{mn} = Cronecker delta

We now consider the other part of eq^n (8)

$$\frac{\sin\theta}{P} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + l(l+1)\sin^2\theta = m^2$$

Dividing throughout by $\sin^2\theta$

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + l(l+1)P - \frac{m^2}{\sin^2\theta} P = 0$$

We transform this equation by putting $x = \cos\theta$

$dx = -\sin\theta \, d\theta$ or $\frac{d}{d\theta} = -\sin\theta \frac{d}{dx}$.

$$\frac{1}{\sin\theta}\left\{-\sin\theta \frac{d}{dx}\left[\sin\theta(-\sin\theta)\frac{dP}{dx}\right]\right\} + l(l+1)P - \frac{m^2}{(1-x^2)}P = 0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right]P = 0 \quad \text{——} \quad (14)$$

SelfStudy.in

Equation (14) is known as Associated Legendre's eq^n. If we put $m=0$ then the eq^n (14) reduces to

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + l(l+1)P = 0 \quad \text{—— (15)}$$

Eq^n (15) is known as Legendre's Equation. Eq^n (15) can be solved by using power series. Let us assume a power series sol^n. $P(x) = \sum_{k=0} a_k x^{k+\lambda}$, $a_0 \neq 0$ —— (16)

Following the techniques used in solving Legendre's Differential Equation we can write the general solution as

memorise

$$P(x) = P_l(x) + Q_l(x) \quad \text{—— (17)}$$

$P_l(x)$ is known as Legendre's Polynomial of the 1st kind and is given by

$$P_l(x) = \sum_{r=0}^{N} \frac{(-1)^r (2l-2r)!\, x^{l-2r}}{2^l\, r!\,(l-r)!\,(l-2r)!}$$

Where $N = \frac{l}{2}$, when l is even; $N = \frac{l-1}{2}$ when l is odd —— (18)

A simple representation of the Legendres polynomials is given by

$$P_l(x) = \frac{1}{2^l\, l!}\frac{d^l}{dx^l}(x^2-1)^l \quad \text{—— (19)}$$

It can also be shown that, the Legendres polynomials form a complete Orthogonal set of function.

Thus the potential in the problem where $m=0$ i.e the problems in which there is Azimuthal symmetry; the potential can be written as

$$V = RS = \left[Ar^{l} + \frac{B}{r^{(l+1)}} \right] \frac{1}{\sqrt{2\pi}} P_l \cos\theta$$

The Solution of Laplaces Equation

$$V = R(r)\, S(\theta) \quad (\phi = \text{const})$$

$$= \sum_{l=0}^{\infty} \left[A_l r^{l} + B_l r^{-(l+1)} \right] \sqrt{\frac{2l+1}{2}}\, P_l (\cos\theta)$$

Because applying the condition of Orthogonality

$$\int_{-1}^{+1} P_l(x)\, P_0(x)\, dx = \frac{2}{2l+1} \delta_{0l}$$

We get, the Orthogonal Legendre's function as:

$$\sqrt{\frac{2l+1}{2}}\, P_l(x).$$