Forced vibration: When a periodic force is applied on a body free to vibrate; in the beginning the vibration is irragular. The body tends to vibrate with its natural frequency; but the driver providing the periodic force; tends to make the body vibrate with its frequency. But with the lapse in time, the natural frequency of vibration of the body dies out; & the body vibrates with the frequency of driver. This is known as forced vibration.

Resonance: Resonance is special case of forced vibration, when the body vibrates with maximum amplitude. It can be shown that resonance occurs when the frequency of the periodic force is same as the natural frequency of vibration of the body.

Let the particle vibrate along x-axis.

m = mass of the particle.

The force acting on the particle are:

(i) The restoring force proportional to displacement with a negative sign: R.

$$\boxed{R.F = -ax}$$

a = Constant = The restoring force per unit displacement

(ii) The damping force, proportional to velocity with a negative sign

$$\boxed{D.F = -b.\frac{dx}{dt}}$$

Where b = Constant. Known as damping force per unit velocity.

(iii) The applied periodic force; known as forcing force $= F \sin \omega t$
Where F is the maximum value of the forcing force.
Applying Newton's law:-

$$m \times \frac{d^2x}{dt^2} = \text{Resultant force} = -ax - b\frac{dx}{dt} + F\sin\omega t.$$

$$\text{or } \frac{d^2x}{dt^2} = -\frac{a}{m}x - \frac{b}{m}\cdot\frac{dx}{dt} + \frac{F}{m}\sin\omega t \quad \text{--- (1)}$$

Put $\frac{a}{m} = \mu^2$ = The restoring force, per unit mass per unit displacement.

$\frac{b}{m} = 2k$ = The damping force per unit mass per unit velocity

$\frac{F}{m} = f$ = The maximum value of the periodic force per unit mass.

Putting these values in equation (1) & rearranging

$$\boxed{\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \mu^2 x = f\sin\omega t} \quad \text{--- (2)}$$

Equation (2) is a second order Non-homogenous differential equation representing the forced vibration of a particle.
Equation (2) has two parts of the solution.
(i) When homogenous part of equation (2) is solved putting the R.H.S equal to zero; it is known as the Complementary Solution.
(ii) When the non-homogenous part of equation (2) is solved, putting the R.H.S equal to $f\sin\omega t$ the part of the solution is known as Particular Integral

The general solution is the sum of the two solutions.

(i) Complementary function:

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \mu^2 x = 0 \quad \text{—— (3)}$$

Equation (3) can be solved by 'D' operator method:

Put $\boxed{\frac{d}{dt} = D}$ & $\boxed{\frac{d^2}{dt^2} = D^2}$

Equation (3) changes to $D^2(x) + 2KD(x) + \mu^2 x = 0$

$$\text{or } [D^2 + 2KD + \mu^2](x) = 0 \quad \text{—— (4)}$$

The bracketed term in equation (4) is quadratic and hence it has two roots:

let m_1 & m_2 be the two roots of the quadratic equation

$$m_1 = -K + \sqrt{K^2 - \mu^2} = -K + m$$
$$m_2 = -K - \sqrt{K^2 - \mu^2} = -K - m$$

Also $(m_1 + m_2) = 2K$ & $m_1 m_2 = \mu^2$ —— (5)

Putting equation (5) in (4):

$$[D^2 - (m_1 + m_2)D + m_1 m_2](x) = 0$$

$$[D(D - m_1) - m_2(D - m_1)](x) = 0$$

or, $(D - m_1)(D - m_2)(x) = 0$

Either $(D - m_1)(x) = 0$ or $(D - m_2)(x) = 0$

$$D(x) = m_1 x \quad \text{or} \quad \frac{dx}{dt} = m_1 x \quad \text{or} \quad \frac{dx}{x} = m_1 dt$$

Integrating both sides;

$\log x = m_1 t + \log A$

or $\boxed{x = A e^{m_1 t}}$ Similarly proceeding with the other solution $(D - m_2)(x) = 0$ we get

$$\boxed{x = B e^{m_2 t}}$$

Hence the general solution of the homogenous equation is the sum of the two solutions.

$\therefore x = Ae^{m_1 t} + Be^{m_2 t}$ —— (6)

where A & B are two unknown constants of integration

SPL Case:

1. If damping is very large; $k >> \mu$

$\therefore m_1 = -k + \sqrt{k^2 - \mu^2} = -k + m$

$m = \sqrt{k^2 - \mu^2} \quad m^2 < k^2 \quad \therefore m < k$

$\therefore m_1 = -k + m = -ve$

$m_2 = -k - m = -ve$

m_1 & m_2 both being negative, 'x' decreases very rapidly with increase in time & vibration ceases very soon.

Case II: In general damping is there but damping is very small. $\boxed{k < \mu}$

$\therefore m = \sqrt{k^2 - \mu^2} = \text{imaginary} = \sqrt{-1(\mu^2 - k^2)} = j\gamma$

where $j = \sqrt{-1}$ & $\gamma = \sqrt{\mu^2 - k^2} = \text{real}$

Equation (6), in this case changes to

$$x = Ae^{(-k+m)t} + Be^{(-k-m)t} = e^{-kt}[Ae^{mt} + Be^{-mt}]$$

or, $x = e^{-kt}[Ae^{j\gamma t} + Be^{-j\gamma t}]$

or, $x = e^{-kt}[A(\cos\gamma t + j\sin\gamma t) + B(\cos\gamma t - j\sin\gamma t)]$

or, $x = e^{-kt}[(A+B)\cos\gamma t + j(A-B)\sin\gamma t]$ ——(7)

Put $A+B = C\sin\delta$ ——(8)

$j(A-B) = C\cos\delta$ ——(9)

where C & δ are two unknowns.

Putting equations (8) & (9) in (7) :-

$$x = e^{-kt}[C\cos\gamma t \cdot \sin\delta + C\sin\gamma t \cdot \cos\delta]$$

or $\boxed{x = Ce^{-kt}\sin(\gamma t + \delta)}$ ——(10)

Equation (10) represents the solution of the complementary function.

(ii) Particular Integral:

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \mu^2 x = f\sin\omega t \text{ ——(2)}$$

Let us assume a trial solution of equation (2)

$$\boxed{x = A_0\sin(\omega t - \phi)} \text{ —— (11)}$$

Where A_0 & ϕ are two unknown constants. Since equation (11) is a solution of equation (2); it must satisfy equation (2) :

$$\frac{dx}{dt} = A_0\omega \cos(\omega t - \phi) \quad \text{———} \quad (12)$$

$$\frac{d^2x}{dt^2} = -A_0\omega^2 \sin(\omega t - \phi) \quad \text{———} \quad (13)$$

Putting equations (11), (12) & (13) in (2) :

$$A_0\left[-\omega^2 \sin(\omega t - \phi) + 2K\omega \cos(\omega t - \phi) + \mu^2 \sin(\omega t - \phi)\right] = f \sin\omega t$$

or $A_0\left[-\omega^2 \sin\omega t.\cos\phi + \omega^2 \cos\omega t.\sin\phi + 2K\omega \cos\omega t.\cos\phi + 2K\omega \sin\omega t.\sin\phi + \mu^2 \sin\omega t.\cos\phi - \mu^2 \cos\omega t.\sin\phi\right] = f\sin\omega t$

or, $A_0\left[\sin\omega t\left\{2K\omega \sin\phi + (\mu^2 - \omega^2)\cos\phi\right\} + \cos\omega t\left\{2K\omega\cos\phi - (\mu^2 - \omega^2)\sin\phi\right\}\right] = f\sin\omega t \quad \text{———} \quad (14)$

Equating the coefficient of $\cos\omega t$ from both sides of equation (14)

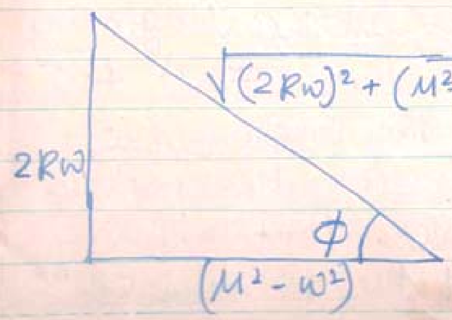
$$A_0\left[2K\omega\cos\phi - (\mu^2 - \omega^2)\sin\phi\right] = 0$$

But $A_0 \neq 0$, Otherwise equation (11) becomes meaningless

$$\therefore 2K\omega\cos\phi - (\mu^2 - \omega^2)\sin\phi = 0$$

$$\text{or } \tan\phi = \frac{2K\omega}{\mu^2 - \omega^2} \quad \text{———} \quad (15)$$

From equation (15), the unknown 'ϕ' can be calculated



$$\therefore \sin\phi = \frac{2K\omega}{\sqrt{(2K\omega)^2 + (\mu^2 - \omega^2)^2}} \quad \text{———} \quad (16)$$

$$\cos\phi = \frac{\mu^2 - \omega^2}{\sqrt{(2K\omega)^2 + (\mu^2 - \omega^2)^2}} \quad \text{———} \quad (17)$$

Equating the coefficient of 'sin ωt' from both sides of equation (14):

$$A_0\left[2k\omega\sin\phi + (\mu^2-\omega^2)\cos\phi\right] = f$$

or $$A_0 = \frac{f}{2k\omega\sin\phi + (\mu^2-\omega^2)\cos\phi}$$

Putting equations (16) & (17): $$A_0 = \frac{f}{\frac{2k\omega.2k\omega}{\sqrt{(2k\omega)^2+(\mu^2-\omega^2)^2}} + \frac{(\mu^2-\omega^2)(\mu^2-\omega^2)}{\sqrt{(2k\omega)^2+(\mu^2-\omega^2)^2}}}$$

or, $$A_0 = \frac{f\sqrt{(2k\omega)^2+(\mu^2-\omega^2)^2}}{(2k\omega)^2+(\mu^2-\omega^2)^2}$$

or, $$A_0 = \frac{f}{\sqrt{(2k\omega)^2+(\mu^2-\omega^2)^2}} \qquad (18)$$

From equation (18) the unknown A_0 can be calculated.

Hence the general solution of forced vibration can be written as the sum of complementary function & the particular integral.

$$\therefore\ x = \boxed{ce^{-kt}\sin(rt+\delta) + A_0\sin(\omega t-\phi)} \qquad (19)$$

From equation (19), we find that, in the beginning the two different frequencies r ($\sqrt{\mu^2-k^2} \simeq \mu$) & ω i.e. the natural frequency (μ) & the frequency (ω) of the forcing force both are present. Hence in the beginning the vibration is complex. But with lapse in time, amplitude of the 1st term ce^{-kt}

decreases & when it becomes negligibly small equation (19) reduces to $\boxed{x = A_0 \sin(\omega t - \phi)}$ — (20)

Then the body vibrates with a single frequency 'ω' the frequency of the forcing force.

Resonance: When the body vibrates with maximum amplitude, the body is said to be resonance with the forcing force.
We now find the condition for resonance:
From (18) the amplitude of the forced vibration

$$A_0 = \frac{f}{\sqrt{(2K\omega)^2 + (\mu^2 - \omega^2)^2}}$$

for A_0 to be maximum we put $\frac{dA_0}{d\omega} = 0$

or $$\frac{dA_0}{d\omega} = f_0\left(-\frac{1}{2}\right)\left[(2K\omega)^2 + (\mu^2-\omega^2)^2\right]^{-3/2}\left[8K^2\omega + 2(\mu^2-\omega^2)(-2\omega)\right] = 0$$

or $$\left(-\frac{1}{2}\right)\left[(2K\omega)^2 + (\mu^2-\omega^2)^2\right]^{-3/2}\left[8K^2\omega - 4\omega(\mu^2-\omega^2)\right] = 0$$

Since both the terms in the 1st bracket, being square & positive their sum can not be zero.

$$8K^2\omega = 4\omega(\mu^2 - \omega^2)$$

or $\mu^2 - \omega^2 = 2K^2$ or $\boxed{\omega_r = \sqrt{\mu^2 - 2K^2}}$ — (21)

Equation (21) gives the resonant frequency which is slightly less than the natural frequency. But if damping is very small K^2 can be neglected compared to μ^2 & $\boxed{\omega_r = \mu}$ i.e the resonance

Occurs when the frequency of the periodic force is same as the natural frequency of vibration of the body.