Free Vibration: When a body vibrates freely without being opposed by any external force of frictional nature or any other force; the vibration is said to be free vibration.

For a particle executing free vibration the only force acting on the particle is the restoring force, proportional to the displacement.

$$\boxed{R.F = -ax}$$

The negative sign indicates that the direction of the force is opposite to the direction of displacement.

a = Const. of proportionality, Known as Restoring force per unit displacement.

let m = mass of the particle.

∴ Equation of motion using Newton's law

$$m.\frac{d^2x}{dt^2} = R.F = -ax \quad \boxed{\text{or } \frac{d^2x}{dt^2} = -\frac{a}{m}.x} \text{ — (1)}$$

Put $\frac{a}{m} = \mu^2$ = Restoring force per unit displacement per unit mass.

Equation (1) can be written as

$\frac{dv}{dt} = -\mu^2 x$ where $v = \frac{dx}{dt}$ = The velocity at the instant of time 't'.

$$\text{or } \frac{dv}{dx}.\frac{dx}{dt} = -\mu^2 x$$

$$\text{or } v.\frac{dv}{dx} = -\mu^2 x$$

Integrating both sides:

$$\boxed{\frac{v^2}{2} = -\mu^2\frac{x^2}{2} + \text{Const}} \text{ — (2)}$$

At the end points : $x = A$ = amplitude ; $v = 0$

from 2 : $0 = -\frac{\mu^2 A^2}{2} + \text{Const}$

$\therefore$ $\text{Const} = \frac{1}{2}\mu^2 A^2$ ——— (3)

Putting (3) in (2) :

$$\frac{1}{2}v^2 = \frac{1}{2}\mu^2 A^2 - \frac{1}{2}\mu^2 x^2$$

$$v^2 = \mu^2 (A^2 - x^2)$$

or $v = \mu\sqrt{A^2 - x^2}$

or $\frac{dx}{dt} = \mu\sqrt{A^2-x^2}$ or $\frac{dx}{\sqrt{A^2-x^2}} = \mu dt$

Integrating both sides:

$$\int \frac{dx}{\sqrt{A^2-x^2}} = \int \mu dt + \text{Const.}$$

Put $x = A\sin\theta$ $\therefore$ $dx = A\cos\theta d\theta$

$$\sqrt{A^2 - x^2} = \sqrt{A^2 - A^2\sin^2\theta} = A\cos\theta$$

$$\therefore \int \frac{dx}{\sqrt{A^2-x^2}} = \int \frac{\cancel{A}\cos\theta\, d\theta}{\cancel{A}\cos\theta} = \theta = \sin^{-1}\frac{x}{A}$$

$\sin^{-1}\frac{x}{A} = \mu t + \phi$ where ϕ is a constant of integration

or, $\boxed{x = A\sin(\mu t + \phi)}$ $\longrightarrow$ Represents the free vibration of a particle with frequency $\mu = 2\pi n$. Hence μ represents the natural frequency of vibration of the particle.