# Magnetohydrodynamics: Or Hydromagnetics:

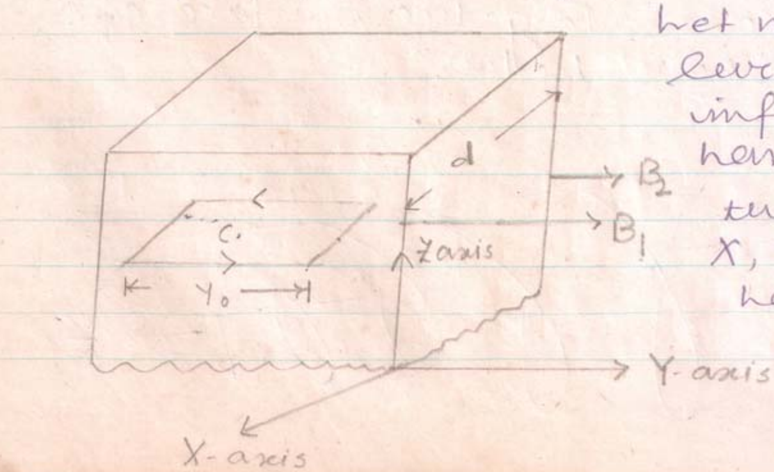
To study the properties of plasma in detail it is difficult to concentrate on the individual contributions made by all the particles. Therefore it is convenient to take measure to a macroscopic approach.

In macroscopic treatment plasma is considered to be divided into small volumes each of which is large compared to the individual particles but small compared to any distance over which the macroscopic property vary appriciably. We can associate with this volume the average values of velocity, magnetic field, density, temp., pressure conductivity etc. appropiate to that volume and the behaviour of plasma is deduced in terms of the action of this volume.

We treat plasma as a conducting fluid and apply simultaneously the laws of electromagnetics & Hydrodynamics and this treatment is known as Magneto-hydrodynamics or Hydromagnetics.

We now obtain the eq$^n$ of hydrodynamics that a plasma must obey:

Let us consider a current sheet; infinite in extent having a very small thickness d. X, Y - Z is a right handed system of axes as shown



The current I in the Sheet is shown to be along Z-axis.
The direction of the magnetic field due to the given direction of current in the Sheet is found to be along Y-axis.
Let $B_1$ and $B_2$ the magnetic induction vector along Y-axis in the front and back faces of the sheet respectively.
Let us consider a rectangular loop C; with sides $\parallel^{l}$ & $\perp^{r}$ to the face of the sheet as shown.
Let $y_0$ be the length of the side of the coil $\parallel^{l}$ to the sheet face of the sheet
Using Ampere's law (circuital law):

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

$$\text{or } \left[ B_1 y_0 + B(d) - B_2 y_0 - B' d \right] = \mu_0 J (y_0 d)$$

Where J is the current density (current per unit area) in the loop.
Since the thickness d is vanishingly small, the 2nd and 4th term may be neglected

$$\therefore \; B_1 y_0 - B_2 y_0 = \mu_0 J d y_0$$

$$\therefore \; B_1 - B_2 = \mu_0 J d$$

let us put $Jd = j_s$ = Surface current density ie current per unit length of the Surface.

$$\therefore \; B_1 - B_2 = \mu_0 j_s \quad — (1)$$

The electrostatic field $\vec{E}$ changes across a charged layer by an amount:

$$E_1 - E_2 = \frac{\sigma}{\epsilon_0} \quad — (2)$$

The force per unit area on the current sheet due to this field is

$$\vec{F} = \int_{-d/2}^{+d/2} (\vec{J} \times \vec{B})\, dx \quad \text{——— (3)}$$

Since $\vec{J}$ and $\vec{B}$ are along Z-axis & Y-axis respectively applying the right hand twel rule for vector product the direction of their product vector $\vec{J} \times \vec{B}$ will be along - X-axis

i.e $\vec{J} \times \vec{B} = [\,|\vec{J}|\,|\vec{B}| \sin 90] \,(-\hat{e}_x)$

Putting in eq$^n$ (3) we have get:

$$\vec{F} = -\hat{e}_x \int_{-d/2}^{+d/2} j\, B\, dx$$

Since $j$ does not change with $x$; but $B$ changes with $x$ ($B = B_1$ at $x = +d/2$ ; $B = B_2$ at $x = -$) is taken out

$$\vec{F} = -\hat{e}_x\, j \left[ \int_{-d/2}^{0} B_2\, dx + \int_{0}^{+d/2} B_1\, dx \right]$$

$$\vec{F} = -\hat{e}_x\, j \left[ B_2 (d/2) + B_1 (d/2) \right]$$

$$\vec{F} = -\hat{e}_x\, j\, d/2\, (B_1 + B_2)$$

from eq$^n$ (1): $\quad B_1 - B_2 = \mu_0\, j\, d$

$$\text{or } j d = \frac{B_1 - B_2}{\mu_0}$$

Putting this value in the above eqⁿ:

$$\vec{F} = -\hat{e}_x \frac{(B_1 - B_2)}{2\mu_0}(B_1 + B_2)$$

$$\text{or } \vec{F} = -\frac{\hat{e}_x}{2\mu_0}(B_1^2 - B_2^2)$$

$$\text{or } \vec{F} = \frac{\hat{e}_x}{2\mu_0}(B_2^2 - B_1^2) \quad — (4)$$

If the only current source under consideration is the sheet itself $\boxed{B_1 = B_2}$ & from eqⁿ (4) :- $\boxed{F = 0}$

Let us now consider the situation in which other current sources are present & $B_1 > B_2$.

Since $B_1 \neq B_2$ $F \neq 0$ & since $B_1 > B_2$ from eqⁿ (4) : $\vec{F} = -\frac{\hat{e}_x}{2\mu_0}(B_1^2 - B_2^2)$ ie force per unit area is along $-x$ direction, which implies that the direction of force (or pressure) is from $\vec{B_1}$ to $\vec{B_2}$. ie from the region of higher induction vector to the region of lower induction vector. If we compare this finding with hydrostatics where we know that the flow of fluid ie force on it acts along higher pressure to lower pressure; we can consider the side where the magnetic field is higher as a region of higher pressure.

Eq$^n$ (4) can then be written as:

$$\vec{F} = \left(P_2^M - P_1^M\right)\hat{e}_x \quad -(5)$$

Where $P_1^M = \dfrac{B_1^2}{2\mu_0}$ & $P_2^M = \dfrac{B_2^2}{2\mu_0}$ are known as magnetic pressure.

Physical Significance: This concept of magnetic pressure was first introduced by Faraday and he thought of tubes of force as elastic filament under tension in the direction of the field and compressed in the transversed direction & latter this concept was mathematically explained by a stressed tensor by Maxwell.

The behaviour of the conducting fluid in an electromagnetic field:

Given: $\vec{J}$ = Current Density within the fluid.

$P$ = The fluid pressure.

The force acting per unit volume of the fluid is:

$$\vec{F} = \vec{J}\times\vec{B} - \nabla P \quad -(6)$$

Equation (6) is known as the fundamental eq$^n$ in Magnetohydrodynamics (MHD).

$$\vec{F} = \vec{J} \times \vec{B} + (-\nabla P) \quad -⑥$$

The (-)ve sign in $\nabla P$ indicates pressure decreases along the direction of $\vec{F}$

In electromagnetics we have if there is no displacement current; $\nabla \times \vec{B} = \mu_0 \vec{J}$

Eq^n ⑥ can be rewritten as:

$$\vec{F} = -\vec{B} \times \vec{J} - \nabla P$$

Putting $\vec{J} = \dfrac{\nabla \times \vec{B}}{\mu_0}$; $\vec{F} = -\vec{B} \times \dfrac{(\nabla \times \vec{B})}{\mu_0} - \nabla P$

$$\text{or } \vec{F} = -\frac{1}{\mu_0}\left[\vec{B} \times (\nabla \times \vec{B})\right] - \nabla P$$

$$\text{or } \vec{F} = -\frac{1}{\mu_0}\left[\nabla\left(\frac{B^2}{2}\right) - (\vec{B}\cdot\nabla)\vec{B}\right] - \nabla P$$

$$\text{or } \vec{F} = +\frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B} - \frac{\nabla B^2}{2\mu_0} - \nabla P$$

$$\text{or } \vec{F} = \frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B} - \nabla\left[P + \frac{B^2}{2\mu_0}\right] \quad -⑦$$

Thus from eq^n ⑦ we can confirm that $\dfrac{B^2}{2\mu_0}$ represents magnetic pressure. "$\dfrac{B^2}{2\mu_0}$" account only for the part of effect of force $\vec{J} \times \vec{B}$. The remaining contribution from this term i.e $\dfrac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B}$ will vanish in the case of a current sheet, and will be present otherwise i.e if there is some extra source. This is so because in case of a current sheet (~~and will be present~~ otherwise ie if there is some extra source) the induction vector $\vec{B}$ being unidirectional the $\nabla \cdot \vec{B}$ will be zero and therefore $\vec{B}$ does not change along field direction. Since the space variation occurs at right angles to $\vec{B}$ therefore $(\vec{B} \cdot \nabla)\vec{B}$ is zero.

If the fluid system is in equilibrium:

$$\vec{F} = 0 = -\nabla\left(P + \frac{B^2}{2\mu_0}\right)$$

ie $$P + \frac{B^2}{2\mu_0} = \text{constant} \quad \text{— (8).}$$

Eqn (8) is the required condition for equilibrium

Pinch Effect: Magnetic Confinement:

We have seen that if we consider a conducting fluid (Plasma) in an electromagnetic field and if $\vec{J}$ is the current density & $P$ is the fluid pressure then the force acting on unit volume of the fluid is:

$$\boxed{\vec{F} = \vec{J} \times \vec{B} - \nabla P}$$

For equilibrium: $\vec{F} = 0$

For a plasma fluid in equilibrium in an electromagnetic field $\nabla P = \vec{J} \times \vec{B}$ — (1)

Taking the scalar product of eqn (1):

$$\vec{B} \cdot \nabla P = \vec{B} \cdot (\vec{J} \times \vec{B})$$