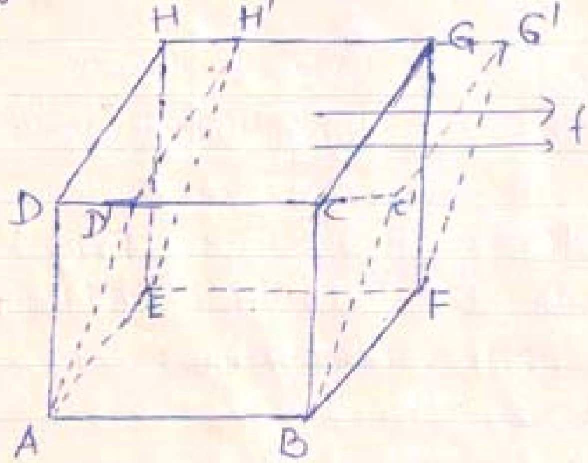
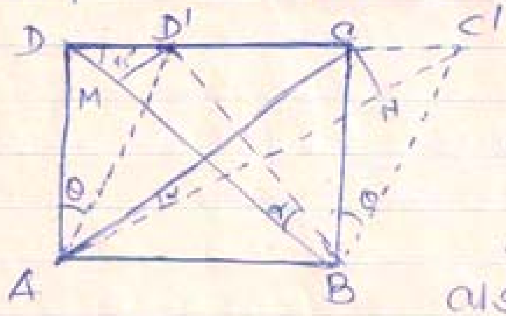
Statement: A Shearing Strain is equivalent to an extensional Strain and an equal Compressional Strain at right angles to each other.

Proof:



Let us consider a cube; with its lower surface ABFE rigidly clamped. A force is applied parallel to the surface DCGH. The cube is sheared. The face ABCD is shown separately below.



Before Shearing; the diagonals AC and BD of the face ABCD are of equal length and they are also at right angles to each other. After shearing; the diagonal AC is extended to AC' and the diagonal BD is shortened to BD'.

From D' and C, D'M and CN are perpendiculars dropped on BD and AC' respectively. Since angles $D'\hat{B}D$ and $C\hat{A}C'$ are very very small say α, from $\triangle D'BM$;

$\cos\alpha = \frac{BM}{BD'}$, $\cos\alpha \simeq 1$ $\therefore \frac{BM}{BD'} \simeq 1$ or $\boxed{BM \simeq B}$

from $\triangle CAN$: $\cos\alpha \simeq 1 = \frac{AN}{AC}$

or $\boxed{AN \simeq AC}$

Compression $= BD - BD' = BD - BM = DM$

Compressional Strain $= \frac{DM}{BD}$

Extensional strain $= \frac{C'N}{AC'} = \frac{C'N}{AC}$

In $\triangle ABD$:

In $\triangle D'DM$; $\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{DM}{DD'}$ $\therefore \boxed{DM = \frac{DD'}{\sqrt{2}}}$ $\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{AD}{BD}$ or $\boxed{BD = \sqrt{2}AD}$

Compressional Strain $= \frac{DD'}{\sqrt{2}\cdot\sqrt{2}AD} = \frac{DD'}{2AD}$

from $\triangle D'AD$; $\tan\theta = \frac{DD'}{AD}$ since θ is very small

$\therefore \tan\theta \simeq \theta$ $\therefore$ $\theta = \frac{DD'}{AD}$

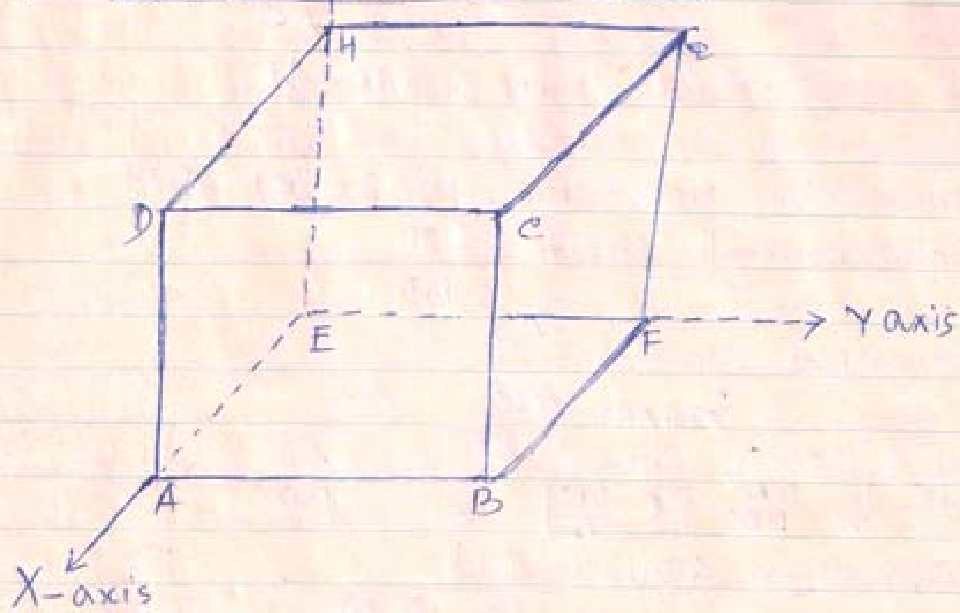
Compressional strain $= \frac{\theta}{2}$

Proceeding exactly in a similar way; it can be shown that; extensional strain $= \frac{\theta}{2}$

Compressional Strain + Extensional strain $= \frac{\theta}{2} + \frac{\theta}{2} = \theta$

$=$ Shearing strain. Proved.

Relation between Elastic Constants:



Let us consider a unit cube. The edges of the cube; are represented by the co-ordinate axes X-Y and Z. Let a stress P be applied

Following the definitions:

Young's modulus $Y = \dfrac{P}{\text{long. strain}}$

$$\text{long. strain} = \frac{P}{Y} \longrightarrow (1)$$

Poissions ratio $\sigma = -\dfrac{\text{lateral strain}}{\text{long. strain}}$

$$\therefore \text{lateral strain} = -\sigma \text{ long. strain} = \frac{-\sigma P}{Y} \longrightarrow (2)$$

We follow a convension: The extensional stress and strain are taken as positive and the compressional stress and strain are taken as negative.

Relation between Y and K : In order to produce a volume strain ; three equal stress +P must be applied simultaneously along three mutually perpendicular directions i.e along three co-ordinate axes, the length of each side of the cube increases.

The increase in length produced along each side of the cube, can be calculated conveniently by considering one stress at a time; and then adding them we get overall change.

Let us write our result in a tabular form :-

Stress applied along			Strain produced along		
X-axis	Y-axis	Z-axis	X-axis	Y-axis	Z-axis
+P	0	0	$+\frac{P}{Y}$	$-\frac{\sigma P}{Y}$	$-\frac{\sigma P}{Y}$
0	+P	0	$-\frac{\sigma P}{Y}$	$+\frac{P}{Y}$	$-\frac{\sigma P}{Y}$
0	0	+P	$-\frac{\sigma P}{Y}$	$-\frac{\sigma P}{Y}$	$+\frac{P}{Y}$
+P	+P	+P	$\frac{P}{Y}(1-2\sigma)$	$\frac{P}{Y}(1-2\sigma)$	$\frac{P}{Y}(1-2\sigma)$

From the last row of the table ; we find that considering the stress +P, simultaneously along three edges ; equal strains produced along the three sides of the cube.

Let e be extension in length, produced along each side of the cube.

$\therefore$ Strain produced along each side $= \frac{e}{l} = \frac{P}{Y}(1-2\sigma)$
(as seen from the table)

$$e = \frac{P}{Y}(1-2\sigma) \rightarrow (3)$$

New length of each side of the cube $= 1+e$

New volume of the cube $= (1+e)^3 = 1+3e+3e^2+e^3$

Since e is very small, higher powers of e can be neglected compared to one.

$\therefore$ The new volume of the cube $= 1+3e$.

$\therefore$ Change in volume of the cube $= 1+3e-1 = 3e$

$$\text{Volume Strain} = \frac{3e}{1} = 3e$$

$$\therefore \text{Bulk modulus } K = \frac{P}{\text{Vol Strain}} = \frac{P}{3e}$$

$$K = \frac{P}{\frac{3P(1-2\sigma)}{Y}}$$

$$\text{or} \quad Y = 3K(1-2\sigma) \longrightarrow (4)$$

Equation (4) gives the relation between Y, K and σ.

Relation between Y and η :-

Let us consider an extensional stress $+P$ and an equal compressional stress $-P$ be applied simultaneously along any two mutually perpendicular edge of the cube, say along X and Y-axes respectively. The strain produced are shown in a tabular form :-

Stress applied along			Strain produced along		
X-axis	Y-axis	Z-axis	X-axis	Y-axis	Z-axis
$+P$	0	0	$+\frac{P}{Y}$	$-\frac{\sigma P}{Y}$	$-\frac{\sigma P}{Y}$
0	$-P$	0	$\frac{-\sigma(-P)}{Y}$	$-P/Y$	$-\sigma(-P)/Y$
Add $+P$	$-P$	0	$\frac{P}{Y}(1+\sigma)$	$-\frac{P}{Y}(1+\sigma)$	0

From last row of the table; we find that when stress +P and −P are considered simultaneously along two perpendicular directions, equal strains are produced along those perpendicular direction and those two equal extensional +ve and compressional −ve strain, at right angles to each other, together is equivalent to a shearing strain θ.

$$\therefore \text{ Shearing Strain } \theta = \frac{P}{Y}(1+\sigma) + \frac{P}{Y}(1+\sigma)$$

$$= \frac{2P}{Y}(1+\sigma)$$

Stress = P

$$\therefore \text{ Rigidity modulus } \eta = \frac{P}{\theta} = \frac{P}{\frac{2P(1+\sigma)}{Y}}$$

$$\text{or } Y = 2\eta(1+\sigma) \quad \text{---} \quad (5)$$

Equation (5) gives the relation between Y, η and σ.

<u>Relation between Y and χ :-</u>

Let us consider a stress $+P$ along X-axis and $+P_1$ along Y-axis and Z axis and the corresponding strains produced are shown in a tabular form.

Stress applied along			Strain produced along.		
X-axis	Y-axis	Z-axis	X-axis	Y-axis	Z-axis
$+P$	0	0	$\frac{P}{Y}$	$-\frac{\sigma P}{Y}$	$-\frac{\sigma P}{Y}$
0	P_1	0	$-\frac{\sigma P_1}{Y}$	$+\frac{P_1}{Y}$	$-\frac{\sigma P_1}{Y}$
0	0	P_1	$-\frac{\sigma P_1}{Y}$	$-\frac{\sigma P_1}{Y}$	$+\frac{P_1}{Y}$
$+P$	$+P_1$	$+P_1$	$\frac{1}{Y}(P-2P_1\sigma)$	$\frac{1}{Y}(P_1-\sigma P-\sigma P_1)$	$\frac{1}{Y}(P_1-\sigma P-\sigma P_1)$

The stress P_1 along Y-and Z-axes have applied to prevent lateral strain, when there should only be longitudinal strain along X-axis due to the stress $+P$.

Hence the value of P_1 must be so chosen that lateral strain along Y and Z axis are zero

i.e $\frac{1}{Y}(P_1 - \sigma P - \sigma P_1) = 0$

$$\sigma P = P_1(1-\sigma)$$

$$P_1 = \frac{\sigma}{1-\sigma} \cdot P \longrightarrow (6)$$

$\therefore$ longitudinal strain produced along X-axis, when there is no lateral strain can be obtained by putting equation (6) in expression for strain along X-axis

$$= \frac{1}{Y}\left[P - 2\sigma \cdot \frac{\sigma P}{1-\sigma}\right]$$

$$= \frac{P}{Y}\left[\frac{1-\sigma-2\sigma^2}{1-\sigma}\right]$$

$$= \frac{P}{Y}\left[\frac{1-2\sigma+\sigma-2\sigma^2}{1-\sigma}\right]$$

$$= \frac{P}{Y}\left[\frac{(1+\sigma)-2\sigma(1+\sigma)}{1-\sigma}\right]$$

$$= \frac{P}{Y}\frac{(1+\sigma)(1-2\sigma)}{1-\sigma}$$

$\therefore$ Axial modulus of elasticity

$$\chi = \frac{\text{Normal Stress}}{\text{Long. Strain . when there is no lateral Strain}}$$

$$= \frac{\cancel{P}}{\frac{\cancel{P}}{Y}\frac{(1+\sigma)(1-2\sigma)}{1-\sigma}}$$

$$Y = \chi\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)} \longrightarrow (8)$$

Equation 8 gives the relation between Y, γ and σ.

Limits of σ:

from equation (4):- $Y = 3K(1-2\sigma)$

* Since Y and K are always positive, the equation (4) will be satisfied only if $(1-2\sigma)$ is +ve.

$$1-2\sigma > 0 \quad \text{ie} \quad 1 > 2\sigma \quad \text{or} \quad \sigma < \frac{1}{2}$$

from equation (5):- $Y = 2n(1+\sigma)$

* Since Y and n are always +ve, the equation (5) will be satisfied only if $1+\sigma$ is +ve ie $1+\sigma > 0$ ie $\boxed{\sigma > -1}$

So $-1 < \sigma < \frac{1}{2}$ [expt: .2 - .4]

---

* $Y = \dfrac{\text{Stress}}{\text{Strain}}$

If stress is extensional +ve strain is also extensional +ve. If the stress is compressional −ve, strain is also compressional so the ratio, modulus of elasticity is always +ve.

Relation between Y, K & σ : $Y = 3K(1-2\sigma)$

" " Y, n & σ : $Y = 2n(1+\sigma)$

" " Y, γ & σ : $Y = \gamma \frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}$