FERMI - DIRAC DISTRIBUTION LAW (OR F-D Statistics):

In this case the particles are identical and indistinguishable but with the additional constraint that no cell can accomodate more than one particle. (This condition is another form of Pauli's exclusion principle) It is required to find the most probable distribution of n_i particles in the g_i cells under the given restriction.

The g_i cells will contain n_i particles and the no. of cells remaining vacant will be $(g_i - n_i)$. Again the g_i cells may be arranged amongst themselves in $g_i!$ ways. But the permutation of n_i particles amongst themselves in $\lfloor g_i$ ways. But the permutation of n_i particles amongst themselves in $n_i!$ way as well as of $(g_i - n_i)$ vacant cells amongst themselves in $\lfloor g_i - n_i$ ways are not going to materially alter the distribution and are therefore totally irrelevant. Thus the most probable distribution is given below

$$\left\{ \frac{g_i!}{n_i!\, g_i - n_i!} \right\}$$

Now, for all the particles
where,

n_1 particles are in g_1 cells having energy u_1 each.

n_2 particles are in g_2 cells having energy u_2 each

etc - - - - - - - - - - etc - - -

The total probability is given by

$$P = GW \quad \text{--- (1)}$$

where,

G = "apriori" probability which is the same for all the cells i.e constant.

and W = Thermodynamic probability.

Then:
$$W = \left[\frac{\lfloor g_1}{\lfloor n_1 \lfloor g_1 - n_1} \cdot \frac{\lfloor g_2}{\lfloor n_2 \lfloor g_2 - n_2} \cdots \frac{\lfloor g_i}{\lfloor n_i \lfloor g_i - n_i} \right]$$

$$W = \prod_i \left[\frac{\lfloor g_i}{\lfloor n_i \lfloor g_i - n_i} \right] \quad \text{--- (2)}$$

From eq^n (1) and (2) we have

$$P = \left[\prod_i \left\{ \frac{\lfloor g_i}{\lfloor n_i \lfloor g_i - n_i} \right\} \times G \right] \quad \text{--- (3)}$$

Taking logarithm of both sides of eq^n (3):
we have,

$$\ln P = \left[\sum \left\{ \ln \lfloor g_i - \ln \lfloor n_i - \ln \lfloor g_i - n_i + \ln G \right] \quad \text{--- (4)}$$

NOW according to the Starlings' approximation

$$\ln \lfloor n_i = n_i \ln n_i - n_i$$

So that eq^n ④ becomes:

$$\ln P = \left[\sum_i \left\{ (g_i \ln g_i - g_i) - (n_i \ln n_i - n_i) - (g_i - n_i) \ln (g_i - n_i) + (g_i - n_i) \right\} + \ln G \right]$$

$$\therefore \ln P = \left[\sum \left\{ (g_i \ln g_i) - (n_i \ln n_i) - (g_i - n_i) \ln (g_i - n_i) \right\} + \ln G \right] \quad \text{--- ⑤}$$

NOW, as the no. of particles is too large a small change in it will not materially affect P.

$$\therefore \delta P = 0$$

So that,

$$\delta (\ln P) = 0 \quad \text{--- ⑥}$$

NOW, differentiating eq^n ⑤ we get

$$\delta (\ln P) = \left[\sum_i \left\{ 0 - n_i \cdot \frac{1}{n_i} \delta n_i - \ln n_i \cdot \delta n_i - (g_i - n_i) \cdot \frac{1}{(g_i - n_i)} (-\delta n_i) - \ln (g_i - n_i) (-\delta n_i) \right\} \right] + 0$$

$$= \left[\sum \left\{ -\delta n_i - \ln n_i - \delta n_i + \delta n_i + \ln (g_i - n_i) \cdot \delta n_i \right\} \right]$$

$$= \sum \left[\left\{ \ln (g_i - n_i) - \ln n_i \right\} \cdot \delta n_i \right] \quad \text{--- (6A)}$$

Hence from eq^n (6) and (6A) we get

$$\delta(\ln P) = \sum_i \left[\left\{ \ln(g_i - n_i) - \ln n_i \right\} \delta n_i \right] = 0 \quad \text{--- (7)}$$

from eq^n (6)

Again the auxillary condition for the conservation of particles gives

$$\sum \delta n_i = 0 \quad \text{--- (8)}$$

While condition for the conservation of energy gives:

$$\sum u_i \delta n_i = 0 \quad \text{--- (9)}$$

Let eq^n (8) and (9) be multiplied by "Lagrangian multipliers" α & β respectively and adding to eq^n (7) we get

$$\left\{ \ln(g_i - n_i) - \ln n_i - \alpha - \beta u_i \right\} \delta n_i = 0 \quad \text{--- (10)}$$

for eq^n (10) to hold the terms in parenthesis must vanish for each value of i. Thus

$$\left\{ \ln(g_i - n_i) - \ln n_i - \alpha - \beta u_i \right\} = 0$$

or $$\ln \frac{g_i - n_i}{n_i} = \alpha + \beta u_i$$

or $$\frac{g_i - n_i}{n_i} = e^{\alpha + \beta u_i}$$

or $$\frac{g_i}{n_i} - 1 = e^{\alpha + \beta u_i}$$

or $$\boxed{n_i = \left[\frac{g_i}{1 + e^{\alpha + \beta u_i}} \right]} \quad \text{--- (11)}$$

$$n_i = \left[\frac{g_i}{e^{\alpha} \cdot e^{\beta u_i} + 1} \right] \quad \text{—— (11)}$$

Eqn (11) gives the "Fermi Dirac distribution law of F-D Statistics".