Fourier Transform:

When the method of separation of variables is applied to certain partial differential equation we often get integral of the form:

$$F(\alpha) = \int_a^b f(x)\, K(\alpha, x)\, dx \quad \text{—— (1)}$$

The function $F(\alpha)$ is said to be the integral transform of the function $f(x)$ by the kernel $K(\alpha, x)$. The following Kernel's are often used in mathematical physics, to carryout transformation

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{i\alpha x}\, dx \quad \text{—— (2)} \quad \text{Fourier transf}$$

$$F(\alpha) = \int_0^{\infty} f(x)\, e^{-\alpha x}\, dx \quad \text{Laplace transform.}$$

$$F(\alpha) = \int_0^{\infty} f(x)\, x J_n(\alpha, x)\, dx \quad \text{— Fourier-Bessel transform.}$$

$$F(\alpha) = \int_0^{\infty} f(x)\, x^{\alpha - 1}\, dx \quad \text{—— Mellian transform.}$$

Theory of Fourier Transforms:

Here we develope the Complex Fourier transform.

The Fourier Series in Complex form, Can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{in\pi x/l} \quad \text{—— (1)} \quad -l \le x \le +l$$

$$\left[f(x) = \frac{a_0}{2} + \sum C_n \left(\cos\frac{n\pi}{l} x + \sin\frac{n\pi}{l} x \right) \right]$$

Multiplying both sides by $e^{-in\pi x/l}\,dx$ & integrating between the limits $-l$ to $+l$.

$$\int_{-l}^{+l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx = \int_{-l}^{+l} \sum_{n=-\infty}^{+\infty} C_n\, e^{\frac{in\pi x}{l}} \cdot e^{-\frac{in\pi x}{l}}\, dx.$$

Or $$\int_{-l}^{+l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx = 2l\, C_n.$$

$$C_n = \frac{1}{2l} \int_{-l}^{+l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx \quad \text{---(2)}$$

To make transition $l \to \infty$; we introduce a new variable 'K' which is defined as $\boxed{K = \frac{n\pi}{l}}$

$$\therefore \Delta n = \frac{\Delta K \cdot l}{\pi} \qquad \text{But } \Delta n = 1 \qquad \boxed{\left(\frac{\Delta K}{\pi}\right) \cdot l = 1}$$

Equation (1) interms of 'K' Can be written as

$$f(x) = \sum_{K=-\infty}^{+\infty} C_n\, e^{iKx}$$

Put $\boxed{C_l(K) = C_n \frac{l}{\pi}}$ $\qquad \therefore \boxed{\begin{aligned} C_n &= \frac{C_l(K)\,\pi}{l} \\ &= C_l(K)\, \Delta K \end{aligned}}$

$$\therefore f(x) = \sum_{-\infty}^{+\infty} C_l(K)\, e^{iKx}\, \Delta K \quad \text{---} (3)$$

Equation (2) in terms of 'K' Can be written as :-

$$C_n = \frac{1}{2l} \int_{-l}^{+l} f(x)\, e^{-iKx}\, dx.$$

or, $\frac{C_n l}{\pi} = \frac{1}{2\pi}\int_{-l}^{+l} f(x)\, e^{-ikx}\, dx$

or $C(k) = \frac{1}{2\pi}\int_{-l}^{+l} f(x)\, e^{-ikx}\, dx$ —— (4).

If we let $l \to \infty$; we get;

from (4): $C(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$ —— (5)

from (2) we get,

$$f(x) = \int_{-\infty}^{+\infty} C(k)\, e^{ikx}\, dk \quad \text{—— (6)}$$

Equations (5) and (6) define Fourier Transform.

Remark: There are various ways of defining Fourier transforms but difference among them the various form are not significant.

The most commonly used form of the Fourier transform is

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\, e^{ikx}\, dx \quad \text{—— (7)}$$

$$\& \; f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(k)\, e^{-ikx}\, dk \quad \text{—— (8).}$$

The above form is obtained by using

$$F(k) = \sqrt{2\pi}\, C(-k).$$