STEFAN'S LAW:

<u>Statement</u>: " The rate of Emission of radiation per unit area of a black body is directly proportional to the fowrth power of its absolute temperature."

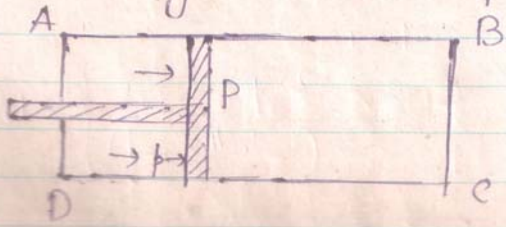
i.e $E = \sigma T^4$

Where σ is known as Stefan's constant.
Boltzmann derived the law from the principles of thermodynamics and consider the net loss of heat by the black body and not radiation by the body alone. Thus according to Boltzmann
" A black body at absolute temp. T surrounded by the another black body at absolute temp. T_0 looses an amount of heat per sec per unit area given by the following relation

$$E = \sigma (T^4 - T_0^4)$$

The above relation is known as Stefan's-Boltzmann law of radiation.

Proof: Consider a cylinderical enclosure ABCD fitted with a closely moving piston (P), moving without friction.



The cylinder has uniform cross-section and is made of perfectly reflecting walls. The

Surface of the piston is also perfectly reflecting.
Let the enclosure be filled with diffused radiation, having an energy density u.
Let the volume of the enclosure be V

Then $U = uV$ — (1)

where U = Total energy of the enclosure.
Let a small amount of heat dQ be given to the system. Then according to the 1st law of Thermodynamics we have

$$dQ = dU + dW \quad \text{—} \quad (2)$$

where,

dU = Small increase in the energy of the system.

& dW = Small amount of work performed by the system.

And,

$$dW = p\,dV \quad \text{—} \quad (3)$$

where,

p = radiation pressure.

& dV = Small increase in volume

Putting from eq^n (1) & (3) in (2) we get

$$dQ = d(uV) + p\,dV$$

$$dQ = u\,dV + V\,du + p\,dV \quad \text{—} \quad (4)$$

But according to Nicholas & Hull, for diffuse radiation,

$$p = \frac{u}{3} \quad \text{—} \quad (5)$$

From eq^n (4) and (5) we have

$$dQ = u\,dV + V\,du + \frac{u}{3}\,dV$$

$$dQ = \left(V\,du + \frac{4}{3}\,u\,dV\right) \quad \text{—} \quad (6)$$

Again from the 2nd law of thermodynamics

$$ds = \frac{dQ}{T} \quad \text{—— (7)}$$

Where,

ds = Small change in entropy of the system at absolute temp T.

from eq^n (6) and (7) we have,

$$ds = \left(\frac{V}{T} du + \frac{4}{3} \frac{u}{T} dV \right) \text{—— (8)}$$

Eq^n (8) shows that 's' is a function of 'u' and 'V'

ie $S = S(u, V)$ — (9)

So that the total differential ds from eq^n (9) is:

$$ds = \frac{\partial s}{\partial u} du + \frac{\partial s}{\partial V} dV \quad \text{—— (10)}$$

Comparing eq^n (8) and (10) we obtain

$$\frac{\partial s}{\partial u} = \frac{V}{T} \quad \& \quad \frac{\partial s}{\partial V} = \frac{4u}{3T} \quad \text{—— (11).}$$

Since ds is a perfect differential we may write,

$$\frac{\partial}{\partial u} \left(\frac{\partial s}{\partial V} \right) = \frac{\partial}{\partial V} \left(\frac{\partial s}{\partial u} \right) \text{—— (12)}$$

using eq^n (11) in (12) we get

$$\frac{\partial}{\partial u} \left(\frac{4u}{3T} \right) = \frac{\partial}{\partial V} \left(\frac{V}{T} \right) \text{—— (13)}$$

Since 'T' is dependent on 'u' and independent of V eq^n (13) becomes

$$\frac{4}{3} \cdot \frac{1}{T} - \frac{4}{3} \cdot \frac{u}{T^2} \cdot \frac{\partial T}{\partial u} = \frac{1}{T}$$

Again from the 2nd law of thermodynamics

$$dS = \frac{dQ}{T} \quad \text{—— (7)}$$

Where,

dS = Small change in entropy of the system at absolute temp T.

from eqn (6) and (7) we have,

$$dS = \left(\frac{V}{T}du + \frac{4}{3}\frac{u}{T}dV\right) \quad \text{—— (8)}$$

Eqn (8) shows that 'S' is a function of 'u' and 'V'

i.e. $S = S(u, V)$ — (9)

So that the total differential dS from eqn (9) is:

$$dS = \frac{\partial S}{\partial u}du + \frac{\partial S}{\partial V}dV \quad \text{—— (10)}$$

Comparing eqn (8) and (10) we obtain

$$\frac{\partial S}{\partial u} = \frac{V}{T} \quad \& \quad \frac{\partial S}{\partial V} = \frac{4u}{3T} \quad \text{—— (11).}$$

Since dS is a perfect differential we may write,

$$\frac{\partial}{\partial u}\left(\frac{\partial S}{\partial V}\right) = \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial u}\right) \quad \text{—— (12)}$$

using eqn (11) in (12) we get

$$\frac{\partial}{\partial u}\left(\frac{4u}{3T}\right) = \frac{\partial}{\partial V}\left(\frac{V}{T}\right) \quad \text{—— (13)}$$

Since 'T' is dependent on 'u' and independent of V eqn (13) becomes

$$\frac{4}{3}\cdot\frac{1}{T} - \frac{4}{3}\cdot\frac{u}{T^2}\cdot\frac{\partial T}{\partial u} = \frac{1}{T}$$

or $\left(\frac{4}{3}-1\right)\cdot\frac{1}{T} = \frac{4}{3}\cdot\frac{u}{T^2}\frac{\partial T}{\partial u}$

$$\frac{1}{\cancel{3}\cancel{T}} = \frac{4}{\cancel{3}}\frac{u}{T^{\cancel{2}}}\frac{\partial T}{\partial u}$$

$$\therefore \frac{\partial u}{u} = \frac{4\,\partial T}{T}$$

Integrating $\ln u = 4 \ln T + \ln A$

where $\ln A$ = constant of Integration

Thus, $u = AT^4$ — (14)

Now,

Rate of readiation of energy per unit area of a black body at an absolute temp. T is proportional to the energy density u.

i.e. $E \propto u$

so that using eq^n (14) we have

$$E \propto T^4$$

Hence, $\boxed{E = \sigma T^4}$ which is Stefan's law.