Q14. Differentiate the following function with respect to x:

$(\log x)^x + x^{\log x}$

Answer: Let $y = (\log x)^x + x^{\log x} \quad \rightarrow (1)$

Now let $y_1 = (\log x)^x$ and $y_2 = x^{\log x}$

$or\ y = y_1 + y_2 \quad \rightarrow (2)$

Differentiating (ii) w.r.to x,

$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} \quad \rightarrow (3)$

Since $y_1 = (\log x)^x$

Taking log on both sides

$\log y_1 = x \log\ (\log x)$

On differentiating w.r.to x

$\frac{1}{y_1}\ \frac{dy_1}{dx} = x \times \frac{1}{\log x} \times \frac{1}{x} + 1 \times \log\ (\log x)$

$or\ \frac{dy_1}{dx} = y_1\ \left(\frac{1}{\log x} + \log\ (\log x)\right)$

$or\ \frac{dy_1}{dx} = (\log x)^x\ \left(\frac{1}{\log x} + \log\ (\log x)\right) \quad \rightarrow (4)$

Also from $y_2 = x^{\log x}$

$\log y_2 = (\log x)\ (\log x) = (\log x)^2$

On differentiating w.r.to x

$\frac{1}{y_2}\ \frac{dy_2}{dx} = 2\ (\log x) \times \frac{1}{x}$

$or\ \frac{dy_2}{dx} = y_2\ \left(\frac{2 \log x}{x}\right) = x^{\log x}\ \left(\frac{2 \log x}{x}\right) \quad \rightarrow (5)$

Putting values of (4) and (5) in equation (3) we obtain

$\frac{dy}{dx} = (\log x)^x\ \left(\frac{1}{\log x} + \log\ (\log x)\right) + x^{\log x}\ \left(\frac{2 \log x}{x}\right)$

Q15. If $y = \log\left[x + \sqrt{x^2 + a^2}\right]$, show that $(x^2 + a^2)\ \frac{d^2y}{dx^2} + x\ \frac{dy}{dx} = 0$

Answer: Considering $y = \log\left[x + \sqrt{x^2 + a^2}\right] \quad \rightarrow (1)$

Differentiating (i) w.r.to x, we get

$\frac{dy}{dx} = \frac{1+\frac{2x}{2\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}$

$or\ \frac{dy}{dx} = \frac{1}{\sqrt{x^2+a^2}} \quad \rightarrow (2)$

From equation (2) we can find $\frac{d^2y}{dx^2}$

This is $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+a^2}} \quad (since\ in\ the\ form\ of\ \frac{u}{v}, \frac{d^2y}{dx^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2})$

$\frac{d^2y}{dx^2} = \frac{\frac{-2x}{2(x^2+a^2)^{\frac{1}{2}}}}{(x^2+a^2)}$

$or\ \frac{d^2y}{dx^2} = -\frac{x}{(x^2+a^2)^{\frac{3}{2}}}$

$or\ (x^2 + a^2)\frac{d^2y}{dx^2} = -\frac{x}{\sqrt{x^2+a^2}} \quad \rightarrow (3)$

Multiplying equation (2) by x and adding equation (3) to get required expression

$(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\frac{x}{\sqrt{x^2+a^2}} + \frac{x}{\sqrt{x^2+a^2}} = 0$

$or\ (x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$