Q17. Evaluate : $\int_{-1}^{2}|x^3 - x|\ dx$

Answer: We know that $x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$

To get rid of mod we can separate the expression into 3 parts between -1 to 2 ranges

$x^3 - x > 0$ where $-1 < x < 0$
$x^3 - x < 0$ where $0 < x < 1$
$x^3 - x > 0$ where $1 < x < 2$

$\therefore \int_{-1}^{2}|x^3 - x|\ dx$

$= \int_{-1}^{0}|x^3 - x|\ dx + \int_{0}^{1}|x^3 - x|\ dx + \int_{1}^{2}|x^3 - x|\ dx$

$= \int_{-1}^{0}(x^3 - x)\ dx - \int_{0}^{1}(x^3 - x)\ dx + \int_{1}^{2}(x^3 - x)\ dx$

$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$

$= 0 - \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2} - 0\right) + 4 - 2 - \left(\frac{1}{4} - \frac{1}{2}\right)$

$= \frac{1}{4} + \frac{1}{4} + 2 + \frac{1}{4} = \frac{11}{4}$

Or, Evaluate : $\int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$

Answer: Considering $I = \int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx \ldots\ldots\ldots (1)$

We know from the properties of definite integral that $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$

Therefore $I = \int_{\mathbf{0}}^{\boldsymbol{\pi}} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx$

or $I = \int_{0}^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx \ldots\ldots\ldots\ldots (2)$

Adding equation (1) and equation (2) we get

$I + I = \int_{0}^{\pi} \frac{(x + \pi - x)\sin x}{1+\cos^2 x}\, dx$

$\Rightarrow\ 2I = \pi \int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$

Putting $\cos x = t\ differentiating - \sin x\, dx = dt$

When $x = 0$ then $t = 1$

When $x = \pi$ then $t = -1$

$\therefore 2I = \pi \int_{1}^{-1} \frac{-dt}{1+t^2} = \pi \int_{-1}^{1} \frac{dt}{1+t^2}$

$= \pi[\tan^{-1} t]_{-1}^{1} = \pi[\tan^{-1} 1 - \tan^{-1}(-1)]$

$= \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$

Therefore $I = \frac{\pi^2}{4}$

Q18. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Answer: The equation of the circles in second quadrant touching coordinate axes is

$(x + a)^2 + (y - a)^2 = a^2 \ldots\ldots\ldots\ldots (1)$

Where coordinate of centre is (-a, a) and radius = a

Which has only one arbitrary constant, a.

Differentiating (1) w.r.t. x

$2(x + a) + 2(y - a)\frac{dy}{dx} = 0$

or $x + a + (y - a)p = 0$ where $p = \frac{dy}{dx}$

or $a = \frac{x+py}{p-1}$

Substituting a in equation (1) we get

$\left(x + \frac{x+py}{p-1}\right)^2 + \left(y - \frac{x+py}{p-1}\right)^2 = \left(\frac{x+py}{p-1}\right)^2$

or $[x(p-1) + x + py]^2 + [y(p-1) - x - py]^2 = (x + py)^2$

or $(x + y)^2p^2 + (x + y)^2 = (x + py)^2$

or $(x + y)^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] = \left(x + y\frac{dy}{dx}\right)^2$

This is our required differential equation.