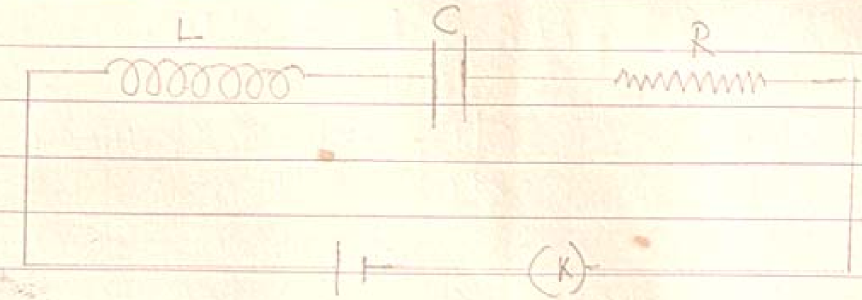
L-C-R circuit :

(i) Charging of a Capacitor through a inductance and resistance :



When the Key (K) is closed, the P.d across the capacitor increases gradually and the charge flows to the plates of the capacitor giving rise to a varying current, till the capacitor gets fully charged i.e the P.d across the capacitor becomes equal to the e.m.f of the source. During this charging the varying current flowing through the circuit, induces an e.m.f across the inductance coil, which according to Lenz's law opposes the cause i.e opposes the growth of charge & hence the growth of charge is delayed.

Analysis:

Given; L = Self inductance of the coil.
C = Capacitance of the capacitor.
R = Resistance in the circuit.
E = The e.m.f of the applied source

Q_0 : The final charge stored in the capacitor.

Let Q be the charge stored in the capacitor, at any instant 't' during charging.
i be the current flowing in the circuit, at that

instant of time t.
The e.m.f eqⁿ of the circuit:

$$L\frac{di}{dt} + \frac{Q}{C} + iR = E. \quad \text{— (1)}$$

But $i = \frac{dQ}{dt}$ $\therefore$ $\frac{di}{dt} = \frac{d^2Q}{dt^2}$ putting in eqⁿ (1):

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E \quad \text{— (2)}.$$

when $Q = Q_0$; $\frac{dQ}{dt} = 0$, $\frac{d^2Q}{dt^2} = 0$ from (2) $\frac{Q_0}{C} = E$

$$Q_0 = EC \quad \text{— (3)}.$$

Re-arranging eqⁿ (2):

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} - \frac{E}{L} = 0$$

or $$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}(Q - EC) = 0$$

putting eqⁿ (3): $$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{(Q - Q_0)}{LC} = 0$$

put $\frac{R}{L} = 2b$ = Const. & $\frac{1}{LC} = K^2$ = Const.

$$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + K^2(Q - Q_0) = 0 \quad \text{— (4)}.$$

put $Q - Q_0 = x$ $\therefore \frac{dQ}{dt} = \frac{dx}{dt}$ & $\frac{d^2Q}{dt^2} = \frac{d^2x}{dt^2}$

putting these values in (4):-

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + K^2x = 0 \quad \text{— (6)}.$$

- Eqⁿ (6) is a homogenous second order differential eqⁿ and can be solved by D-operator method but $\frac{d}{dt} = D$ & $\frac{d^2}{dt^2} = D^2$

$\therefore D^2(x) + 2bD(x) + K^2(x) = 0$

or $[D^2 + 2bD + K^2](x) = 0$ — 7.

The bracketed term being quadratic, has two roots. Let m_1 and m_2 be the roots

$m_1 = -b + \sqrt{b^2 - K^2} = -b + m$

where $m = \sqrt{b^2 - K^2}$

$m_2 = -b - \sqrt{b^2 - K^2} = -b - m.$

We know that $m_1 + m_2 = -2b$ & $m_1 m_2 = K^2$ — 8

putting eqn 8 in 7:

$[D^2 - (m_1 + m_2)D + m_1 m_2](x) = 0$

or $[D(D - m_1) - m_2(D - m_1)](x) = 0$

or $(D - m_1)(D - m_2)(x) = 0$

either $(D - m_1)x = 0$ or $(D - m_2)(x) = 0$

or $D(x) = m_1 x$ or $\frac{dx}{dt} = m_1 x$ or $\frac{dx}{x} = m_1 dt$

Integrating both sides: $\log x = m_1 t + \log A$ — 9

where $\log A$ is unknown constant of integration

$\therefore x = Ae^{m_1 t}$

Similarly proceeding with the other solution

$x = B.e^{m_2 t}$, B is a const. of integration.

Hence the general solution of eqn 6:

$x = Ae^{m_1 t} + Be^{m_2 t}$ — 9

putting the value of 'x' from eq^n (5) :

$$Q - Q_0 = Ae^{m_1 t} + Be^{m_2 t}$$

$$Q = Q_0 + Ae^{(-b+m)t} + Be^{(-b-m)t}$$

$$\text{or } Q = Q_0 + e^{-bt}\left[Ae^{mt} + Be^{-mt}\right] \quad \text{— (10)}$$

Eq^n (10) gives the charge stored in the capacitor at any instant of time 't' during charging.
Differentiating eq^n (10) w.r.to 't' we get $i = \frac{dQ}{dt}$

$$i = \frac{dQ}{dt} = 0 + (-b)e^{-bt}\left[Ae^{mt} + Be^{-mt}\right] + e^{-bt}\left[mA - mBe^{-mt}\right]$$

$$\text{or } i = me^{-bt}\left[Ae^{mt} - Be^{-mt}\right] - be^{-bt}\left[Ae^{mt} + Be^{-mt}\right] \text{ —}$$

We now evaluate the unknown constants A and B by applying initial condition:

(i) at $t=0$, $Q=0$ Hence from (10) :

$$0 = Q_0 + [A\cdot 1 + B\cdot 1] \quad \therefore A + B = -Q_0 \quad \text{— (12)}$$

(ii) at $t=0$, $i=0$ from eq^n (11) :

$$0 = m\cdot 1\left[A\cdot 1 - B\cdot 1\right] - b\cdot 1\left[A\cdot 1 + B\cdot 1\right]$$

$$\text{or } A - B = \frac{b}{m}(A+B) = -\frac{b}{m}Q_0 \quad \text{— (13)}$$ | putting eq^n (12)

$$\left.\begin{aligned} A + B &= -Q_0 \\ A - B &= -\frac{b}{m}Q_0 \end{aligned}\right\}$$ Adding eq^n (12) and (13) :

$$A = -\frac{Q_0}{2}\left(1 + \frac{b}{m}\right) \quad \text{— (14)}$$

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Substracting eq^n (13) from (12):

$$B = -\frac{Q_0}{2}\left(1 - \frac{b}{m}\right) \quad -(15)$$

Thus the unknown constants A & B are known from eq^n (14) and (15).

We now analyse the growth of charge in the capacitor in few special cases:

For the given values of R, L & C we have clearly three possibilities :-

e^{mt}

(i) $\frac{R^2}{4L^2} > \frac{1}{LC}$ (ii) $\frac{R^2}{4L^2} = \frac{1}{LC}$ (iii) $\frac{R^2}{4L^2} < \frac{1}{LC}$

-(11)

Case I: Let $\frac{R^2}{4L^2} > \frac{1}{LC}$ i.e $b^2 > k^2$

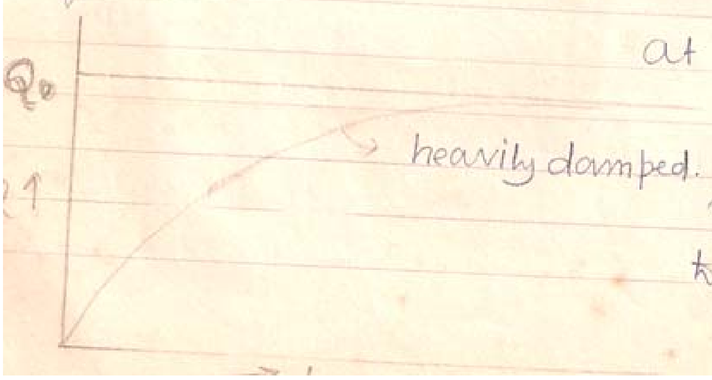
Then $m = \sqrt{b^2 - k^2}$ is real & $m < b$

$m_1 = -b + m = -ve.$

$m_2 = -b - m = -ve.$

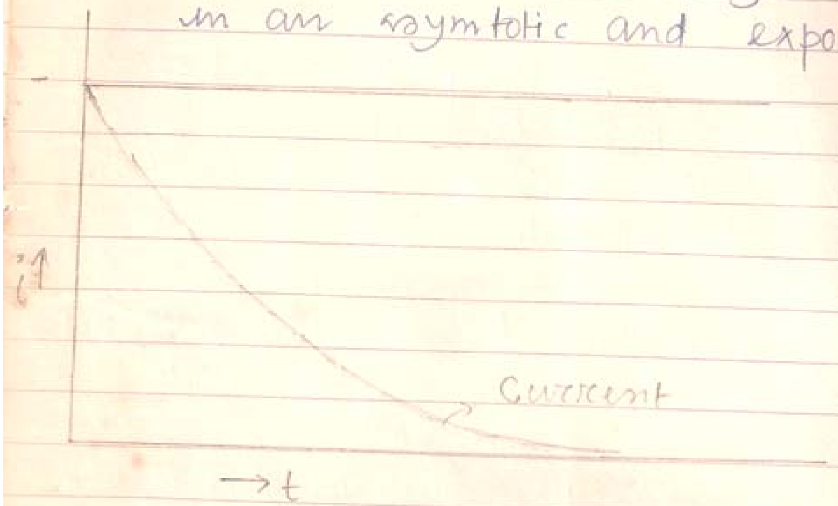
Hence from (10) $Q = Q_0 + [A e^{m_1 t} + B e^{m_2 t}]$

m_1 and m_2 both being negative both the terms in the bracket decreases rapidly with increase in time & Q reaches to its final value Q_0 in a steady fashion.



at $t = \infty$; both the terms in the bracket vanishes and $Q = Q_0$. Thus the charge grows to its final value in an exponential and asymptotic fashion. In this case the growth of charge is said to be heavily damped.

From Eq^n (11) at $t=\infty$ both the terms vanishes and $i=0$; thus the current falls to zero in an asymptotic and exponential fashion



Case II: Let $\frac{R^2}{4L^2} = \frac{1}{LC}$ or $b^2 = k^2$; $m=0$

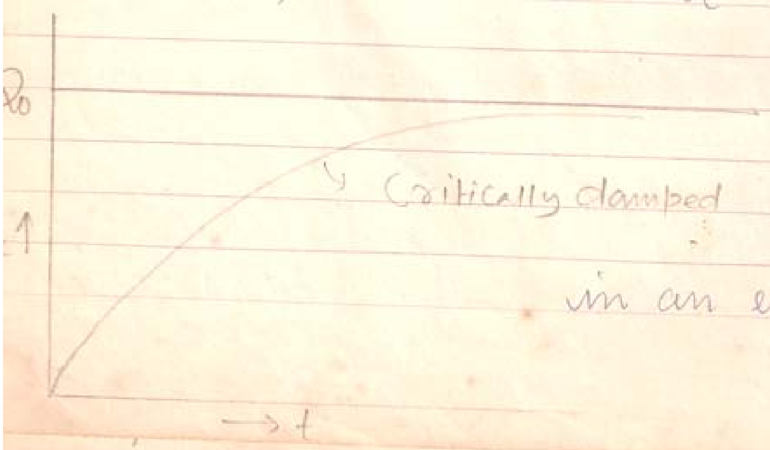
from (10) $Q = Q_0 + e^{-bt}[A+B]$

putting eq^n (12):

$$Q = Q_0 - Q_0 e^{-bt}$$

$$Q = Q_0(1-e^{-bt}) \quad \text{--- (16)}$$

at $t=\infty$; $e^{-\infty}=0$ $\therefore$ $Q = Q_0$



Thus in this case the charge grows to its final value in an exponential fashion

and is asymptotic to the line $Q = Q_0$.
In this case only one damping term being present the charge grows to its final value in minimum time and the growth of charge is said to be critically damped.

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Case III : Let $\frac{R^2}{4L^2} < \frac{1}{LC}$ i.e. $b^2 < K^2$

$m = \sqrt{b^2 - K^2} = \sqrt{-1(K^2 - b^2)} = j\omega$

where $\omega = \sqrt{K^2 - b^2}$ = real $[\omega^2 + b^2 = K^2]$

In this case eq^n (10) can be written as;

$$Q = Q_0 + e^{-bt}[Ae^{j\omega t} + Be^{-j\omega t}]$$

or $Q = Q_0 + e^{-bt}[A(\cos\omega t + j\sin\omega t) + B(\cos\omega t - j\sin\omega t)]$

or $Q = Q_0 + e^{-bt}[(A+B)\cos\omega t + j(A-B)\sin\omega t]$

putting eq^n (12) and (13) :

$$Q = Q_0 + e^{-bt}\left[-Q_0\cos\omega t + j\left(-Q_0\frac{b}{m}\right)\sin\omega t\right]$$

or $Q = Q_0 - Q_0 e^{-bt}\left[\cos\omega t + j\cdot\frac{b}{j\omega}\cdot\sin\omega t\right]$

or $Q = Q_0 - \frac{Q_0 e^{-bt}}{\omega}[\omega\cos\omega t + b\sin\omega t]$

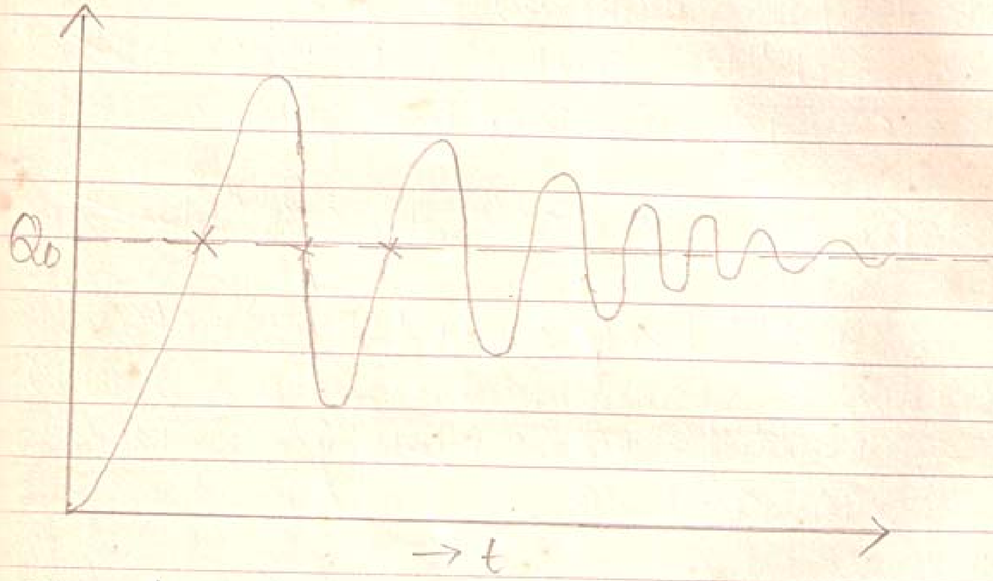
put $\omega = K\sin\alpha$ & $b = K\cos\alpha$

$\therefore Q = Q_0 - \frac{Q_0 e^{-bt}}{\sqrt{K^2 - b^2}}[K\sin\alpha\cdot\cos\omega t + K\cos\alpha\sin$

or $Q = Q_0 - \frac{Q_0 K e^{-bt}}{\sqrt{K^2 - b^2}}\sin(\omega t + \alpha)$ — (17)

Eqⁿ (17) gives the growth of charge when $\frac{R^2}{4L^2} < \frac{1}{LC}$. Since eqⁿ (17) represents an oscillatory motion, the charge in the capacitor grows to its final value Q_0 in an oscillatory fashion.

The amplitude of oscillation $\frac{Q_0 K e^{-bt}}{\sqrt{K^2 - b^2}}$ decreases exponentially with time due to the factor e^{-bt}.



The frequency of oscillation is $f = \frac{\omega}{2\pi}$

$$\therefore f = \frac{1}{2\pi}\sqrt{K^2 - b^2}$$

$$\text{or } f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \text{—— (18)}$$

Eqⁿ (18) is known as the natural frequency of oscillation of L-C-R circuit.

Differentiating eqⁿ (17) w.r.t. t we get; the current flowing through the circuit at the instant t.

$$Q = Q_0 - \frac{Q_0 k e^{-bt}}{\sqrt{k^2-b^2}} \sin(\omega t + \alpha) \quad \text{— (17)}$$

$$i = \frac{dQ}{dt} = 0 - \frac{Q_0 k}{\omega}\left[(-b)e^{-bt}\sin(\omega t+\alpha) + e^{-bt}\omega\cos(\omega t+\alpha)\right]$$

$$\text{or } i = \frac{Q_0 k e^{-bt}}{\sqrt{k^2-b^2}}\left[b\sin(\omega t+\alpha) - \omega\cos(\omega t+\alpha)\right]$$

put $b = k\cos\alpha$ & $\omega = k\sin\alpha$ $\quad b^2+\omega^2 = k^2$

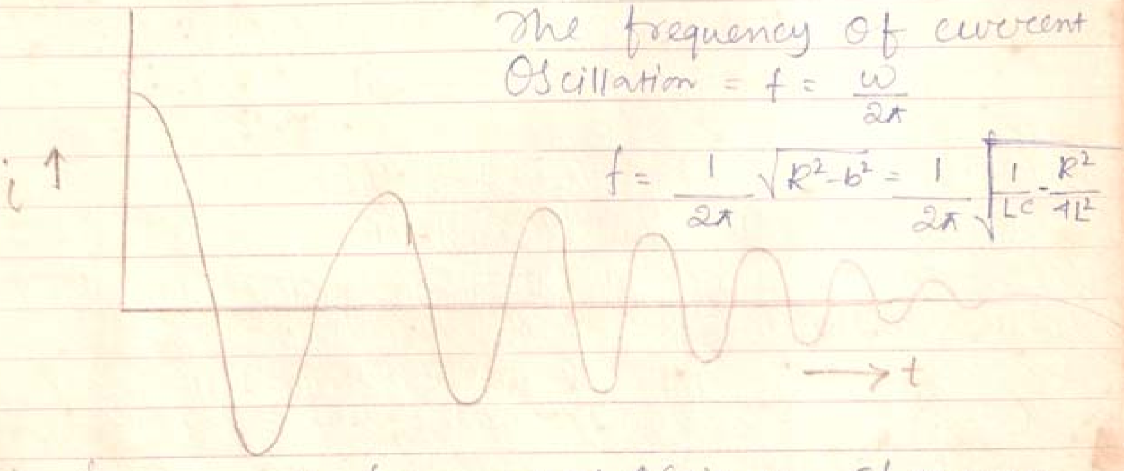
$\omega^2 = k^2 - b^2 \quad \omega = \sqrt{k^2-b^2}$

$$\therefore i = \frac{Q_0 k e^{-bt}}{\sqrt{k^2-b^2}}\left[k\cos\alpha \cdot \sin(\omega t+\alpha) - k\sin\alpha \cdot \cos(\omega t+\alpha)\right]$$

$$\text{or } i = \frac{Q_0 k^2 e^{-bt}}{\sqrt{k^2-b^2}} \sin\{(\omega t+\alpha) - \alpha\}$$

$$\text{or } i = \frac{Q_0 k^2 e^{-bt}}{\sqrt{k^2-b^2}} \sin\omega t \quad \text{— (19)}$$

From eq^n (19) we find that the current flowing in the circuit is also oscillatory in fashion; the amplitude of oscillation decreases due to the factor, e^{-bt}, and finally current becomes zero.



The frequency of current oscillation $= f = \frac{\omega}{2\pi}$

$$f = \frac{1}{2\pi}\sqrt{k^2-b^2} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

f is same as the frequency of oscillation of charge.