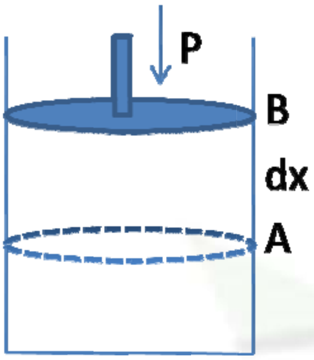
Relation between Cp and Cv: To prove Cp-Cv = R

Let us consider 1 kg mole of a perfect gas enclosed in a cylinder fitted with a piston at constant pressure. Some heat is supplied to the gas externally the temperature as well as the volume of the gas rises.



Given:

P = constant pressure applied on the piston

α = Area of cross section of the piston

ΔT= rise in temperature of the gas

dx = The distance AB through which the piston is pushed up

dv = α dx = The increase in volume of the gas.

Cp & Cv = Molar specific heat capacities of the gas at constant pressure and constant volume respectively.

Q be the heat supplied.

Although both the changes take place simultaneously but for convenience of calculation we assume them to be occurring in steps first the temperature changes and then volume changes.

Step 1: The volume is assumed to be constant and the temperature rises by dT and

Let Q1 be the heat required for that

$Q1 = 1.\ Cv.dT$ Joules $\longrightarrow$ (1)

Step II : The volume of the gas then increases by pushing the piston up by a distance dx against a force P x α

Let Q2 = energy required to do the mechanical work in pushing the piston up.

$Q2 = P \times (\alpha \times dx) = P \times dv$ Joules $\longrightarrow$ (2)

Since Q1 and Q2 are the part of the total heat supplied to the gas Q hence

$Q = Q1 + Q2 \longrightarrow (3)$

But if we look at the change as a whole we find that Q heat rises the temperature of

1 kg mole of the gas by dT at constant pressure.

$Q = 1.\ Cp.dT \longrightarrow (4)$

Putting equation (1), (2) and (4) in (3)

$CpdT = CvdT + Pdv$

$Cp - Cv = P.dv/dT \longrightarrow (5)$

For perfect gas the equation is $PV = nRT$

For 1 kg Mole n = 1 $PV = RT$

Differentiating with P constant $P\ dv/dT = R \longrightarrow (6)$

Putting equation (6) in (5) we get

$Cp - Cv = R \longrightarrow (7)$

If M be the molecular weight of the gas and cp & cv are the ordinary specific heat of the gas at constant pressure and constant volume respectively then

$Mcp - Mcv = R$

$cp - cv = R/M = \gamma$

Transformation of Heat: The state of a gas can be represented by three variables

(i) Pressure

(ii) Volume

(iii) Temperature

If one of these changes the state of the gas changes.

Isothermal Transformation: When the state of a gas is changed keeping its temperature constant the change is known as Isothermal change.

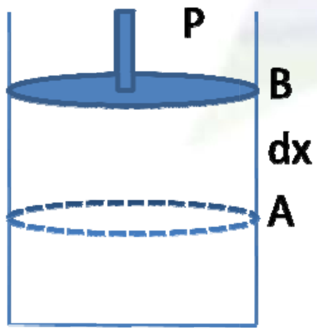
For Isothermal change $\Delta T = 0$

Generally slow changes are treated as Isothermal. In isothermal process the pressure and volume of a gas is related by PV= constant when temperature is constant (Boyles law).

Adiabatic Process: When the state of a gas is changed so that no heat is allowed to leave the system or no heat is allowed to enter into the system from the surrounding i.e. exchange of heat energy between the system and the surrounding is zero the change is known as Adiabatic process. Thus if the system is thermally insulated from the surrounding the change is adiabatic $\Delta Q=0$. Generally sudden changes are treated as adiabatic.

Relation between pressure, volume and temperature of a gas in adiabatic transformation:

Let us consider 1 kg mole of a perfect gas enclosed in a cylinder fitted with a piston. If some heat is supplied to the gas the temperature as well as volume of the gas increases.



Let ΔQ = Amount of heat supplied to the gas

P = Pressure applied on the piston.

α = Area of cross section of the piston

dT = Rise in temperature of the gas

dx = AB = The distance through which the piston is pushed up

$dv = \alpha\, dx$ =The increase in volume of the gas

Cp & Cv = Molar specific heat capacities of the gas at constant pressure and constant volume respectively.

Although both the changes take place simultaneously but for convenience of calculation we assume them to be occurring in steps.

Step 1: Let the volume remain constant and the temperature rises by dT only and let Q1 be the amount of heat absorbed by the gas for this change

Q1 = 1.Cv.dT Joules ⟶ (1)

Step II: Temperature remains constant at T + dT and the volume of the gas increases by dv

Let Q2 = Amount of heat energy absorbed by gas in doing the mechanical work, in pushing the piston up by a distance dx against a force (Pxα)

$$\therefore Q_2 = (P \times \alpha)dx = P \times (\alpha \times dx) = Pdv \text{ Joules} \rightarrow (2)$$

Since both the parts of heat absorbed are obtained from the supplied heat

Since both parts of the heat absorbed from supplied heat

$$\Delta Q = Q_1 + Q_2$$

$$\Delta Q = C_v dT + PdV$$

For adiabatic transformation $\Delta Q = 0$

$$C_v dT + PdV = 0 \rightarrow (3)$$

For a perfect gas $PV = nRT$ (here $n = 1$)

$$PV = RT$$

Differentiating : $PdV + VdP = RdT$

$$dT = \frac{PdV + VdP}{dT} = \frac{PdV + VdP}{Cp - Cv} \rightarrow (4)$$

Putting equation (4) in (3) :

$$C_v \frac{PdV + VdP}{Cp - Cv} + PdV = 0$$

$$C_v Vdp + C_p PdV = 0$$

Dividing both sides by $C_p PV$

$$\frac{dP}{P} + \frac{C_p}{C_v}\frac{dV}{V} = 0$$

$$\frac{dP}{P} + \gamma \frac{dV}{V} = 0 \qquad \text{Where } \gamma = \frac{C_p}{C_v} = \text{Constant}$$

Integrating both sides:

$$\int \frac{dP}{P} + \int \gamma \frac{dV}{V} = \int 0$$

$$\log_e P + \gamma \log_e V = \text{constant}$$

$$\log_e PV^\gamma = \text{constant}$$

Taking antilog $PV^\gamma = e^{\text{constant}} = \text{constant}$

$$PV^\gamma = \text{constant} \rightarrow (5)$$

We know that $PV = RT$

$$P = \frac{RT}{V} \rightarrow (6)$$

$$V = \frac{RT}{P} \rightarrow (7)$$

From equation (5) & (6)

$$\frac{RT}{V} V^\gamma = \text{constant}$$

$$TV^{\gamma\text{-}1} = \text{constant} \rightarrow (8)$$

From equation (5) and (7)

$$P\left(\frac{RT}{P}\right)^\gamma = \text{constant}$$

$$T^\gamma P^{1-\gamma} = \text{constant} \rightarrow (9)$$